CHAPTER 3: Problem Solving and Reasoning

Objectives:
a. Differentiate inductive and deductive reasoning
b. Understand the Polya's Problem Solving Strategy
c. Apply different problem solving strategy in solving patterns, and recreational activities.

Lesson 1: Inductive and Deductive Reasoning

Inductive Reasoning

- Is the process of getting a general conclusion by observing the specific examples or set.

Example 1: Use inductive reasoning to predict a next number.

$$3, 6, 9, 12, 15, ?$$

Solution: Each successive number is 3 larger than the preceding number. Thus, we predict that the next number that 3 larger than 15 is 18.

Example 2: Use inductive reasoning to predict a next number.

$$1, 3, 6, 10, 15, ?$$

Solution: The first two numbers differ by 2. The second and third numbers differ by 3. It appears that the difference between any two numbers is always 1 more than the preceding difference. Since 10 and 15 is differ by 5 we predict that the next in the list will be 6 larger than 15 which is 21.

Example 3: Use inductive reasoning to make a conjecture.

Consider the following procedure: Pick a number. Multiply the number by 8, add 6 to the product, divide the sum by 2, and subtract 3.

Complete the above procedure for several different numbers. Use inductive reasoning to make a conjecture about the relationship between the size of the resulting number and the size of the original number.

Solution: Suppose we pick 5 as our original number. Then the procedure would produce the following results:

Original Number: 5
Multiply by 8: $8 \times 5 = 40$
Add 6: $40 + 6 = 46$

Divide by 2: $46 \div 2 = 23$
Subtract 3: $23 - 3 = 20$

We started with 5, and by following the procedure we got 20 as the result. Starting at 6 as our original number produce a result of 24. Starting with 10 produces a final result of 40. Starting with 100 produces a final result of 400. In each of these cases the resulting number is 4 times the original number. We conjecture that by following the given procedure produces 4 times the original number.

Try to discuss the following scenarios as an example of Inductive reasoning:

Scenario 1: Jennifer always leaves for school at 7:00 a.m. Jennifer is always on time. Jennifer assumes, then, that if she leaves at 7:00 a.m. for school today, she will be on time

Scenario 2: The cost of goods was $1.00. The cost of labor to manufacture the item was $0.50. The sales price of the item was $5.00. So, the item always provides a good profit for the stores selling it.

Scenario 3: Ray is a football player. All the other football players on the high school team weigh more than 170 pounds. Therefore, Ray must weigh more than 170 pounds.

Deductive Reasoning

- Is the process of reaching a conclusion by general assumption, procedures or principle. It is distinguish from inductive reasoning since deductive reasoning is finding conclusion by applying general principle and procedure in the observation.

Example 1: Use deductive reasoning to establish a conjecture.

Use deductive reasoning to show that the following procedure produces a number that is four times the original number.

Procedure: Pick a number. Multiply the number by 8, add 6 to the product, divide the sum by 2, and subtract 3

Solution:

Let *n* represent the original number.

Multiply the number by 8:	$8n$
Add 6 to the product:	$8n+6$
Divide the sum by 2:	$\frac{8n+6}{2}=4n+3$
Subtract 3:	$4n+3-3=4n$

Observe the result in the procedure, we started as n and ended up with $4n$ The procedure given in this example produces a number that is four times the original number.

Deductive reasoning gives us hint that: A is equal to B. B is also equal to C. Given those two statements, you can conclude A is equal to C using deductive reasoning. To summarize; $a=b$, $b=c$ combining two expression using deductive reasoning it means that $a=c$.

Try to apply deductive reasoning into $a=b$, $b=c$ and $a=c$ in other examples:

Example 1: All numbers ending in 0 or 5 are divisible by 5. The number 35 ends with a 5, so it must be divisible by 5.

Example 2: All birds have feathers. All robins are birds. Therefore, robins have feathers.

Example 3: Elephants have cells in their bodies, and all cells have DNA. Therefore, elephants have DNA.

Inductive vs. Deductive Reasoning

Try to analyze the following examples to understand the difference between Inductive reasoning and Deductive reasoning.

Examples:

a. During the past 10 years an apple tree has produced plums every other year. Last year the tree did not produce apple, so this year the tree will produce apples.

b. All home improvements cost more than the estimate. The contractor estimated that my home improvement will cost Php 35,000. Thus, my home improvement will cost more than Php 35, 000.

Solutions:

a. This agreement reaches a conclusion by observing the specific examples, so it is an example of inductive reasoning.

b. Because the conclusion is a specific case of a general assumption, the argument is an example of deductive reasoning.

For more information about inductive and deductive reasoning, please click the link provided.

- https://study.com/academy/lesson/inductive-and-deductive-reasoning.html
- https://www.khanacademy.org/math/algebra-home/alg-series-and-induction/alg-deductive-and-inductive-reasoning/v/deductive-reasoning-1

Proof

- The old, colloquial meaning of "prove" is: Test, try out, determine the true state of affairs.

The trouble is, "mathematical proof has two meanings:

1. the practical meaning, is informal, imprecise. Practical mathematical proof is what we do to make each other believe our theorems. It's argument that convinces the qualified, skeptical expert

2. theoretical mathematical proof, is formal. Aristotle helped make it. This is supposed to be a "formalization, idealization, rational reconstruction of the idea of proof (P. Ernest, private communication).

Problem A: What does meaning number 1 have to do with meaning number 2?
Problem B: How come so few notice Problem A Is it uninteresting? Embarrassing?
Problem C: Does it matter?

Problem C is easier than A and B. It matters, morally, psychologically, and philosophically

Scenario: When you're a student, professors and books claim to prove things. But they don't say what's meant by "prove." You have to catch on. Watch what the professor does, then do the same thing. Then you become a professor, and pass on the same "know-how" without "knowing what" that your professor taught you.

Two interesting Assertions

Assertion 1: Logicians don't tell mathematicians what to do. They make a theory out of what mathematicians actually do.

Assertion 2: Any correct practical proof can be filled in to be a correct theoretical proof.

- "If you can do it, then do it!" "It would take too long. And then it would be so deadly boring, no one would read it." Assertion B is commonly accepted. Yet I've seen no practical or theoretical argument for it, other than absence of counter-examples. It may be true. It's a matter of faith.

Example: "Two points determine a line, and two lines determine a point, unless the lines are parallel." Projective geometry brings in ideal points at infinity—one point for each family of parallel lines. The axiom becomes: "Two points determine a line, and two lines determine a point." This is "right."

Mathematical Intuition

- Mathematical intuition is coming across a problem, glancing at it, and using your logical instinct to pull out an answer without asking further questions.

Example:

If you look at a number set with a range between 53 and 73 , comprising of the numbers 53, 64, 63, 61, 65, 73and were asked to calculate the average: your first guess would be around the 63-65 mark.

How?

There is one number below the $60s = 53$

There is one number above the $60s = 73$

These two numbers virtually cancel each other out.

Every other number is in the $60s$, with $61, 63$, 64 and 65, so the average must be around a 63.

All of this happens in less than a second - this is mathematical intuition.

Certainty

- In math, this one is something that is accurate and absolute.

Even if it's granted that the need for certainty is inherited from the ancient past, and is religiously motivated, its validity is independent of its history and its motivation.

Example:

We take three examples. First, good old

$$2+2=4$$

Second,

The angle sum of any triangle equals two right angles." Finally, a more sophisticated example: a convergent infinite series. Label the first example

Formula A: $2+2=4$

By the associative law of addition, Formula A then is:

$$1+1+1+1=1+1+1+1$$

For more knowledge about Intuition, Proof and Certainty, please click the link provided.

- https://www.uni-siegen.de/fb6/phima/lehre/phima13/quellentexte/seminar_-_hersh/hersh-chapter4.pdf

REMEMBER

- **Inductive Reasoning** is the process of getting a general conclusion by observing the specific examples or set.
- **Deductive Reasoning** is the process of reaching a conclusion by general assumption, procedures or principle.
- **Proof** The old, colloquial meaning of "prove" is: Test, try out, determine the true state of affairs.
- **Mathematical intuition** is coming across a problem, glancing at it, and using your logical instinct to pull out an answer without asking further questions.
- **Certainty** is something that is accurate and absolute.

ACTIVITY: Written in a Picture

In this activity, students get to look at various pictures of real-life things and draw a general conclusion based off of the observations they make.

Procedure:

1.) Students work individually or with a partner.

2.) Students make observations based off of pictures they see to come to a general conclusion.

Lesson 2: Problem Solving Strategies

George Polya was a Hungarian who immigrated to the United States in 1940. His major contribution is for his work in problem solving.

George Polya created his famous **four-step process for problem solving,** which is used all over to aid people in problem solving:

1. Understand the problem.
2. Devise a plan. (Translate)
3. Carry out the problem. (Solve)
4. Look back. (Check and interpret)

Example 1: Twice the difference of a number and 1 is 4 more than that number. Find the number.

Step 1: Understand the problem.

Make sure that you read the question carefully several times.

Since we are looking for a number, we will let

***x* = a number**

Step 2: Devise a plan.

Twice the difference of a number 1 and 4 more than that number

$$2(x-1) = x+4$$

Step 3: Carry out the problem. (Solve)

$$2(x-1)=x+4$$
$$2x-2=x+4$$
$$2x-2-x=x+4-x$$
$$x-2=4$$
$$x=4+2$$
$$x=6$$

Step 4: Look back. (Check and interpret)

If you take twice the difference of 6 and 1, that is the same as 4 more than 6, so this does check. The final answer is 6.

Example 2: Carol has written a number pattern that begins with 1, 3, 6, 10, 15. If she continues this pattern, what are the next four numbers in her pattern?

Step 1: Understand the problem.

You need to find 4 numbers after 15.

Step 2: Devise a plan.(Translate)

You can find a pattern. Look at the numbers. The new number depends upon the number before it.

Step 3: Carry out the problem. (Solve)

Look at the numbers in the pattern.

$3 = 1 + 2$ (starting number is 1, add 2 to make 3)

$6 = 3 + 3$ (starting number is 3, add 3 to make 6)

$10 = 6 + 4$ (starting number is 6, add 4 to make 10)

$15 = 10 + 5$ (starting number is 10, add 5 to make 15)

Step 4: Look back. (Check and interpret)

New numbers will be

$$15+6=21$$
$$21+7=28$$
$$28+8=36$$
$$36+9=45$$

Problem Solving Strategies

- These are the different problem solving strategies that you can use in Mathematics.

Problem Solving Strategies
• Look for a pattern
• Make an organized list
• Guess and Check
• Make a table
• Work backwards
• Use logical reasoning
• Draw a diagram
• Solve a simpler problem
• Read the problem carefully
• Create problem solving journals

- **Look for a pattern**

Example: Find the sum of the first 100 even positive numbers.
Solution:

The sum of the first 1 even positive numbers is $2 \ or 1(1+1) = 1(2)$.

The sum of the first 2 even positive numbers is $2 + 4 = 6 \ or 2(2+1) = 2(3)$.

The sum of the first 3 even positive numbers is
$2 + 4 + 6 = 12 \ or 3(3+1) = 3(4)$.

The sum of the first 4 even positive numbers is
$2 + 4 + 6 + 8 = 20 \ or 4(4+1) = 4(5)$.

Look for a pattern:
The sum of the first 100 even positive numbers is $2 + 4 + 6 + ... = ?$
or $100(100+1) = 100(101) \ or \ 10{,}100$.

- **Make an organized list**

Example: Find the median of the following test scores:
$73, 65, 82, 78, \textit{and } 93.$
Solution: Make a list from smallest to largest:
65
73
78 *Since* 78 *is the middle number, the median is* 78.
82
93

- **Guess and check**

Example: Which of the numbers $4, 5, \textit{or } 6$ is a solution to
$(n + 3)(n - 2) = 36?$
Solution: Substitute each number for "n" in the equation. Six is the solution since $(6 + 3)(6 - 2) = 36$.

- **Make a table**

Example: How many diagonals does a $13-gon$ have?
Solution: Make a table:

Number of sides	Number of diagonals
3	0
4	2
5	5
6	9
7	14
8	20

Look for a pattern. Hint: If n is the number of sides, then n(n-3)/2 is the number of diagonals. Explain in words why this works. A $13-gon$ would have $\frac{13(13-3)}{2} = 65$ diagonals.

- **Work backwards**

Example: Fortune Problem: a man died and left the following instructions for his fortune, half to his wife; $\frac{1}{7}$ of what was left went to his son; $\frac{2}{3}$ of what was left went to his butler; the man's pet pig got the remaining $\$2000$. How much money did the man leave behind altogether?

Solution: The pig received $\$2000$.

$\frac{1}{3}$ of $? = \$2000$

$? = \$6000$

$\frac{6}{7}$ of $? = \$6000$

$? = \$7000$

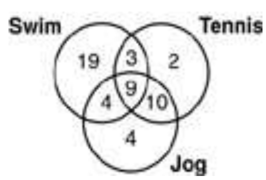
$\frac{1}{2}$ of $? = \$7000$

$? = \$14,000$

- **Use logical reasoning**

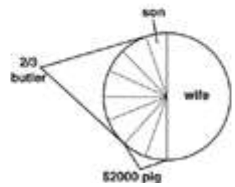
Example: At the Keep in Shape Club, 35 people swim, 24 play tennis, and 27 jog. Of these people, 12 swim and play tennis, 19 play tennis and jog, and 13 jog and swim. Nine people do all three activities. How many members are there altogether?

Solution: Hint: Draw a Venn Diagram with 3 intersecting circles.



- **Draw a diagram**

Example: Fortune Problem: a man died and left the following instructions for his fortune, half to his wife; $\frac{1}{7}$ of what was left went to his son; $\frac{2}{3}$ of what was left went to his butler; the man's pet pig got the remaining $\$2000$. How much money did the man leave behind altogether?



- **Solve a simpler problem**

Example: In a delicatessen, it costs $\$2.49$ for a half pound of sliced roast beef. The person behind the counter slices 0.53 pound. What should it cost?

Solution: Try a simpler problem. How much would you pay if a half pound of sliced roast beef costs $\$2$ and the person slices 3 pounds? If a half pound costs $\$2$, then one pound would cost $2 \times \$2 \; or \; \$4$. Multiply by the number of pounds needed to get the total: $3 \times \$4 = 12.$
Now try the original problem: If a half pound costs $\$2.49$, then one pound would cost $2 \times \$2.49 \; or \; \$4.98.$ Multiply by the number of pounds needed to get the total: $.53 \times \$4.98 = \$2.6394 \; or \; \$2.64.$

- **Read the problem carefully**
Know the meaning of all words and symbols in the problem.

Example: List the ten smallest positive composite numbers.
Solution: Since positive means greater than 0 and a composite number is a number with more than two whole number factors, the solution is $4, 6, 8, 9, 10, 12, 14, 15, 16, 18.$
For example, 4 has three factors, $1, 2, \; and \; 4.$

Sort out information that is not needed.

Example: Last year the Williams family joined a reading club. Mrs. Williams read 20 books. Their son Jed read 12 books. Their daughter Josie read 14 books and their daughter Julie read 7 books. How many books did the children of Mr. and Mrs. Williams read altogether?
Solution: You do not need to know how many books Mrs. Williams has read since the question is focusing on the children.

Determine if there is enough information to solve the problem.

Example: How many children do the Williams have?
Solution: There is not enough information to solve the problem. You do not know if Josie, Julie, and Jed are the only children.

- **Create problem solving journals**
  Students record written responses to open-ended items such as those tested on FCAT in mathematics. Student identifies problem solving strategies.
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Mathematical Problems Involving Patterns

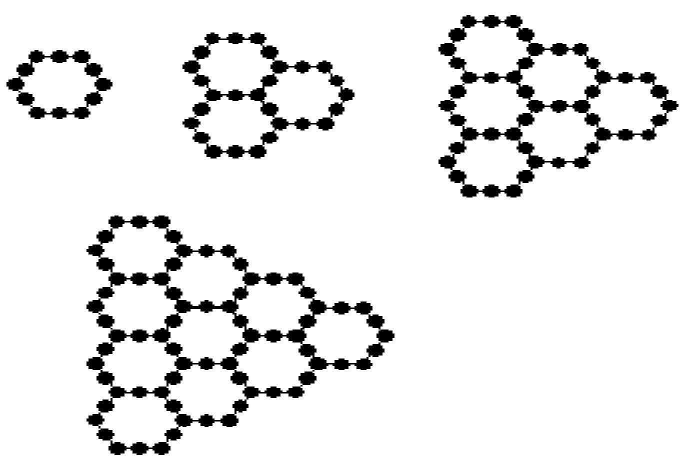
In solving mathematical problems involving patterns, you need to have an analyzation on how does a problem be solve.

Some are examples of problems involving patterns:

Example 1:

Each hexagon below is surrounded by 12 dots.

a) Find the number of dots for a pattern with 6 hexagons in the first column.
b) Find the pattern of hexagons with 229 dots.



Solution:

1st column	Pattern	Total dots
1	12	12
2	12 + 16	28
3	12 + 16 + 21	49

4	12 + 16 + 21 + 26	75
5	12 + 16 + 21 + 26 + 31	106
6	12 + 16 + 21 + 26 + 31 + 36	**142**
7	12 + 16 + 21 + 26 + 31 + 36 + 41	183
8	12 + 16 + 21 + 26 + 31 + 36 + 41 + 46	229

a) The number of dots for a pattern with 6 hexagons in the first column is 142
b) If there are 229 dots then the pattern has 8 hexagons in the first column.

Example 2:

Each member of a club shook hands with every other member who came for a meeting. There were a total of 45 handshakes. How many members were present at the meeting?

	A	B	C	D	E	F	G	H	I	J
A										
B	●									
C	●	●								
D	●	●	●							
E	●	●	●	●						
F	●	●	●	●	●					
G	●	●	●	●	●	●				
H	●	●	●	●	●	●	●			
I	●	●	●	●	●	●	●	●		
J	●	●	●	●	●	●	●	●	●	
HS	9	8	7	6	5	4	3	2	1	

Solution:

$$Total = 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 45 \ handshakes$$

There were 10 members.

- For more puzzles in Mathematics, please click the link provided: https://www.mathsisfun.com/puzzles/index.html

Recreational problems using Mathematics

Puzzles and riddles are perhaps the most well-known activities within recreational math. Math puzzles and riddles are fun and interesting, and they help improve problem solving skills and thinking capacity! Puzzles and riddles are also an important area of research for many mathematicians.

The Puzzle: Alphabet Numbers

Using any letter only once, what are the largest and smallest numbers that you can write down in words?

Example: EIGHTY
But not NINETY as N is used twice
Bonus 1: allow negatives such as MINUS TWO
Bonus 2: allow calculations such as TWO SQUARED

The Puzzle: Alphabet Numbers **(SOLUTION)**

Largest: FIVE THOUSAND
Smallest: ZERO or NOUGHT

Bonus 1: allow negatives
Smallest: MINUS FORTY

Bonus 2: allow calculations
Largest: SIXTY CUBED (=216,000) from Steven Nguyen

The Riddle 1: How do you go from 98 to 720 using just one letter?

The Riddle 1: Answer

Add an "x" between "ninety" and "eight". Ninety x Eight = 720

The Riddle 2: There is a three digit number. The second digit is four times as big as the third digit, while the first digit is three less than the second digit. What is the number?

The Riddle 2: Answer

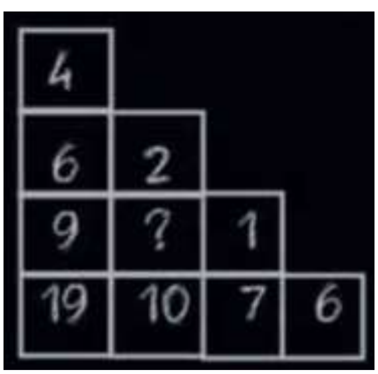
141

For more riddles in Mathematics, please click the link provided:
- https://www.brainzilla.com/brain-teasers/riddles/math/

REMEMBER

- Polya's 4 step problem solving strategies are:
  - Understand the problem.
  - Devise a plan
  - Carry out the problem
  - Look back

ACTIVITY: You got me.
Choose your partner, and try to answer these 5 riddles

1. How can you add eight 8's to get the number 1,000? (only use addition).
2. How do you make the number 7 an even number without addition, subtraction, multiplication or division?
3. How can you take 2 from 5 and leave 4?
4. I am a number with a couple of friends, quarter a dozen, and you'll find me again. What am I?
5. 

http://web.mnstate.edu/peil/M110/Worksheet/PolyaProblemSolve.pdf
https://nzmaths.co.nz/problem-solving-strategies
https://study.com/academy/lesson/inductive-and-deductive-reasoning-