CHAPTER 7: Mathematical Graphs

Objectives:
a. Identify Mathematical Graphs
b. Differentiate Euler Circuits and Paths; the weighted graphs
c. Use Graph coloring in application of Map Coloring.

Lesson 1: Graphs and Euler Circuits

In the modern world, planning efficient routes is essential for business and industry, with applications as varied as product distribution, laying new fiber optic lines for broadband internet, and suggesting new friends within social network websites like Facebook.

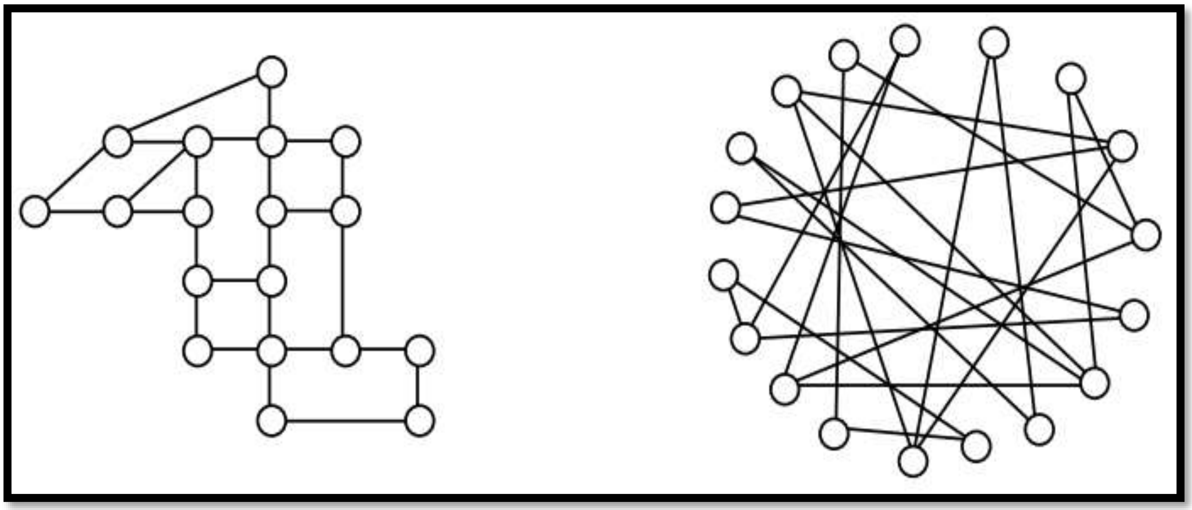
This field of mathematics started nearly 300 years ago as a look into a mathematical puzzle (we'll look at it in a bit). The field has exploded in importance in the last century, both because of the growing complexity of business in a global economy and because of the computational power that computers have provided us.

Graph

- Is a set of points called **vertices** and line segments or curves called **edges** that connect vertices. Graphs can be used to represent different scenarios.
- consists of a set of dots, called **vertices**, and a set of **edges** connecting pairs of vertices.

Example:

While we drew our original graph to correspond with the picture we had, there is nothing particularly important about the layout when we analyze a graph.

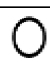
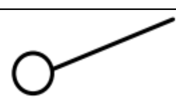
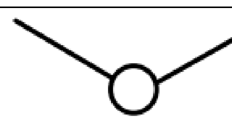
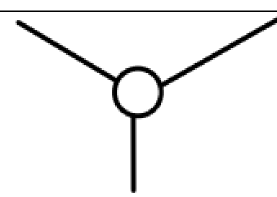
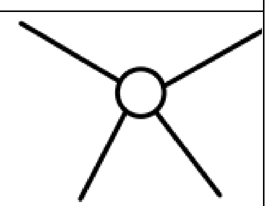


- **VERTEX**

A vertex is a dot in the graph that could represent an intersection of streets, a land mass, or a general location, like “work” or “school”. Vertices are often connected by edges. Note that vertices only occur when a dot is explicitly placed, not whenever two edges cross. Imagine a freeway overpass—the freeway and side street cross, but it is not possible to change from the side street to the freeway at that point, so there is no intersection and no vertex would be placed.

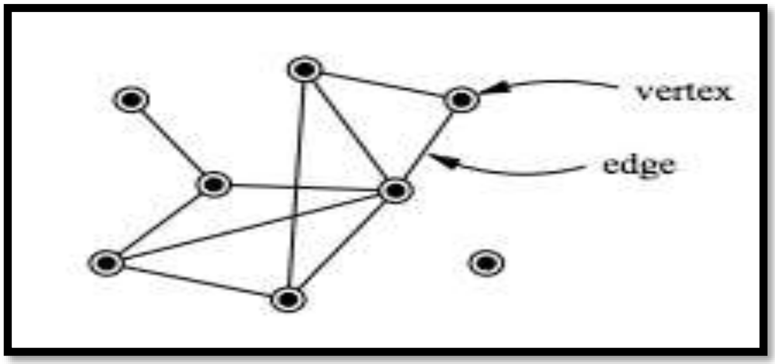
DEGREE OF A VERTEX

The degree of a vertex is the number of edges meeting at that vertex. It is possible for a vertex to have a degree of zero or larger.

Degree 0	Degree 1	Degree 2	Degree 3	Degree 4
				

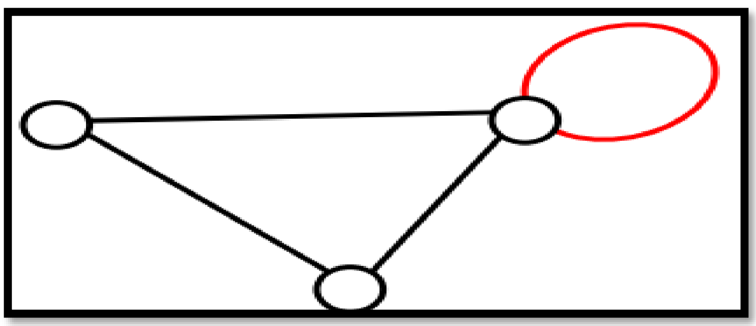
- **EDGES**

Edges connect pairs of vertices. An edge can represent a physical connection between locations, like a street, or simply that a route connecting the two locations exists, like an airline flight.



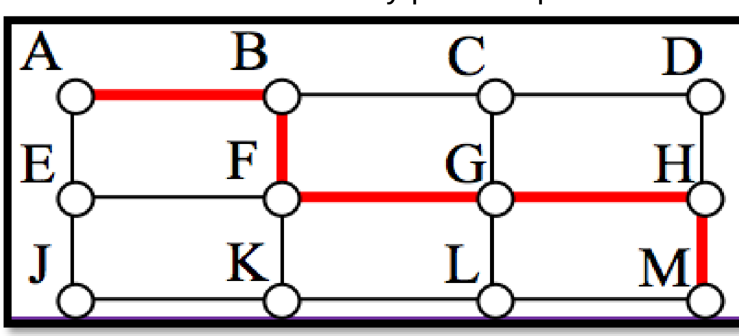
- **LOOP**

  A loop is a special type of edge that connects a vertex to itself. Loops are not used much in street network graphs.



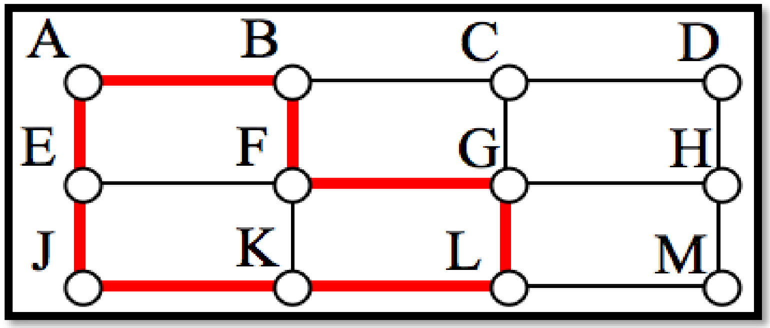
- **PATH**

  A path is a sequence of vertices using the edges. Usually we are interested in a path between two vertices. For example, a path from vertex A to vertex M is shown below. It is one of many possible paths in this graph.



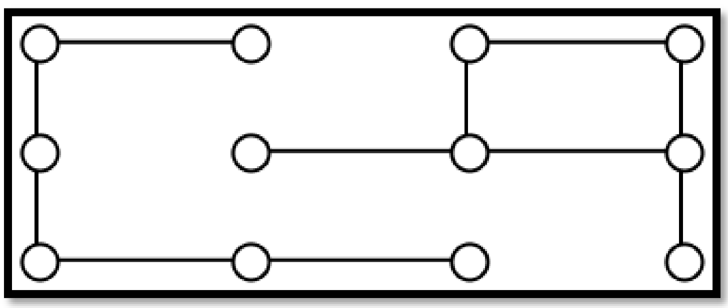
- **CIRCUIT**

A circuit is a path that begins and ends at the same vertex. A circuit starting and ending at vertex A is shown below.



- **CONNECTED**

  A graph is connected if there is a path from any vertex to any other vertex. Every graph drawn so far has been connected. The graph below is disconnected; there is no way to get from the vertices on the left to the vertices on the right.



- **WEIGHTS**

  Depending upon the problem being solved, sometimes weights are assigned to the edges. The weights could represent the distance between two locations, the travel time, or the travel cost. It is important to note that the distance between vertices in a graph does not necessarily correspond to the weight of an edge.

Example: Construct a Graph

The following table lists five students at a college. An "x" indicates that the two students participate in the same study group this semester.

	Matt	Amber	Oscar	Laura	Kyla

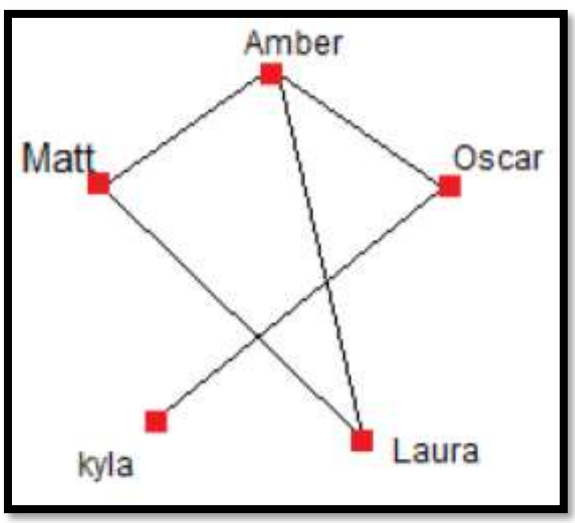
Matt	___	X		X	
Amber	X	___	X	X	
Oscar		X	___		X
Laura	X	X		___	
Kyla			X		___

a. Draw a graph that represents this information where each vertex represents a student and an edge connects two vertices. If the corresponding students study together.

b. Use your graph to answer the following question: Which student is involved in the most study groups with the others? Which student has only one study group in common with the others? How many study groups does Laura have in common with the others?

Solution:

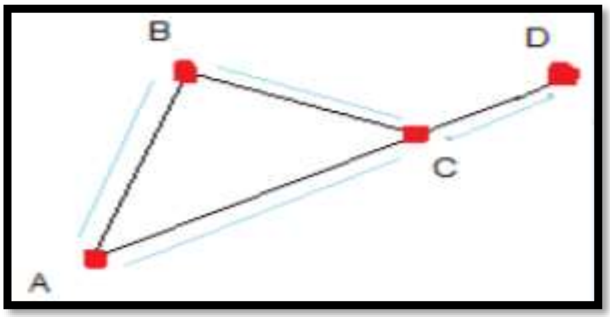
a. We draw five vertices (in any configuration we wish) to represent the five students and connect vertices with edges according to table.

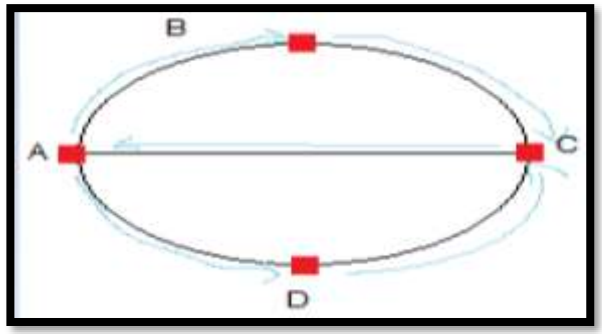


b. The vertex corresponding to Amber is connected to more edges than the others, so he is involved with more study groups (three) than the others. Kyla is the only student with one study group in common, as her vertex is the only one connected to just one edge. Laura's vertex is connected to two edges, so she shares two study groups with the others.

Euler Path

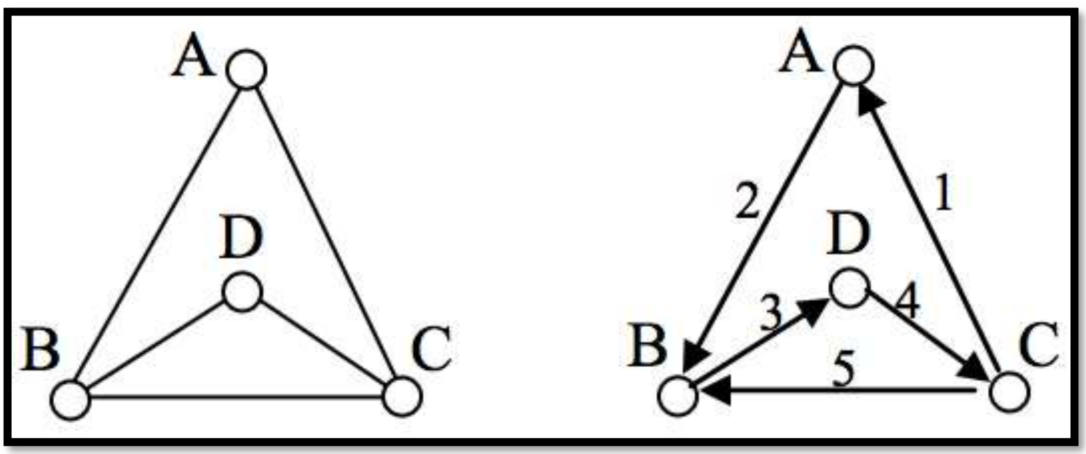
- is a path that uses every edge in a graph with no repeats. Being a path, it does not have to return to the starting vertex.





Example:

In the graph shown below, there are several Euler paths. One such path is CABDCB. The path is shown in arrows to the right, with the order of edges numbered.

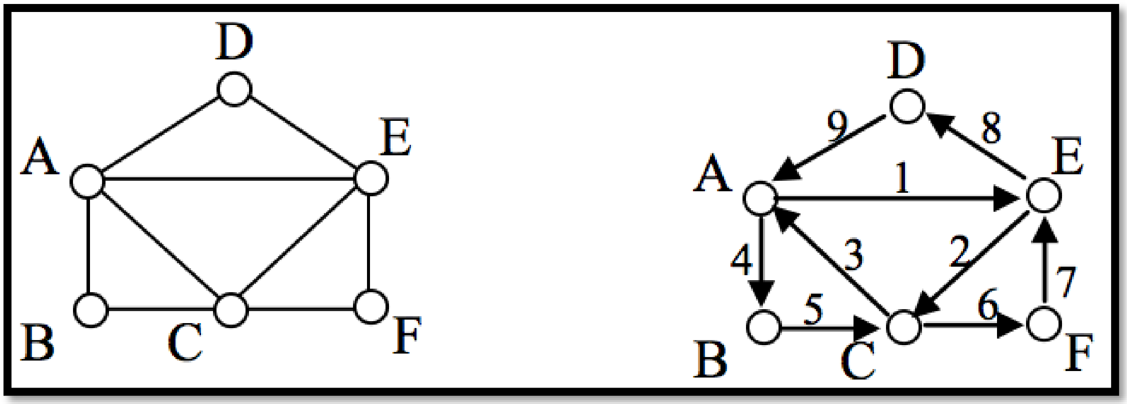


Euler Circuit

- is a circuit that uses every edge in a graph with no repeats. Being a circuit, it must start and end at the same vertex.

Example:

The graph below has several possible Euler circuits. Here's a couple, starting and ending at vertex A: ADEACEFCBA and AECABCFEDA. The second is shown in arrows.



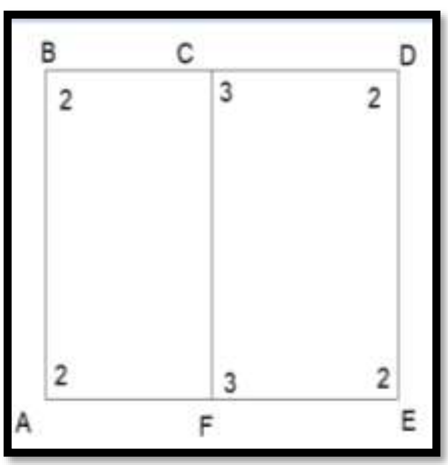
EULER'S PATH AND CIRCUIT THEOREMS

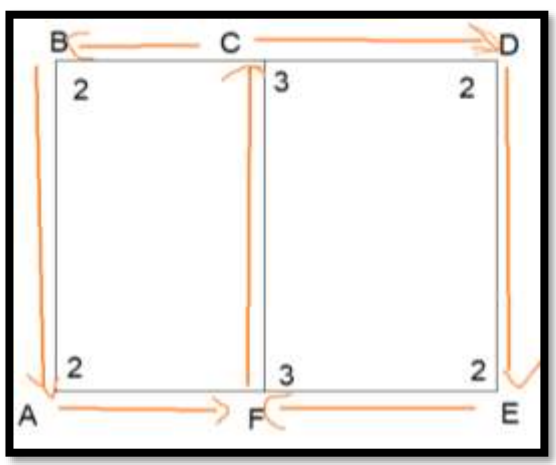
1. A graph will contain an Euler path if it contains at most two vertices of odd degree.
2. A graph will contain an Euler circuit if all vertices have even degree.

Note: These theorems do not tell us how to find the Euler circuit and path, it just say that it exist.

Example: Determine if each graph has a Euler path. If has one, find the Euler path.

A.



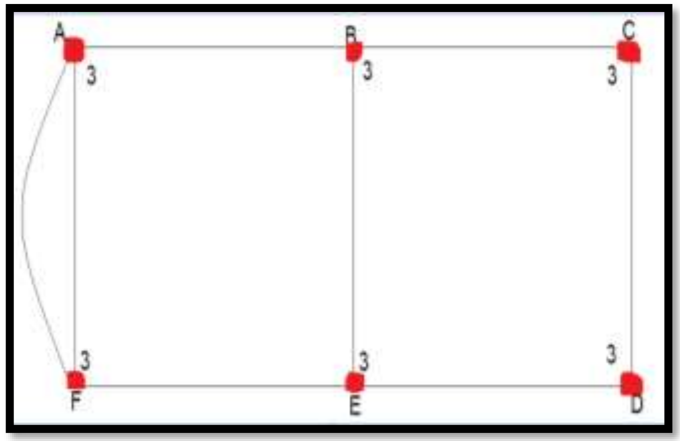


From the Euler Path theorem, it says that a graph will contain Euler path if it contains at most two vertices of ODD DEGREE.

It means that this graph has a Euler path, and the Euler path is CDEFCBAF.

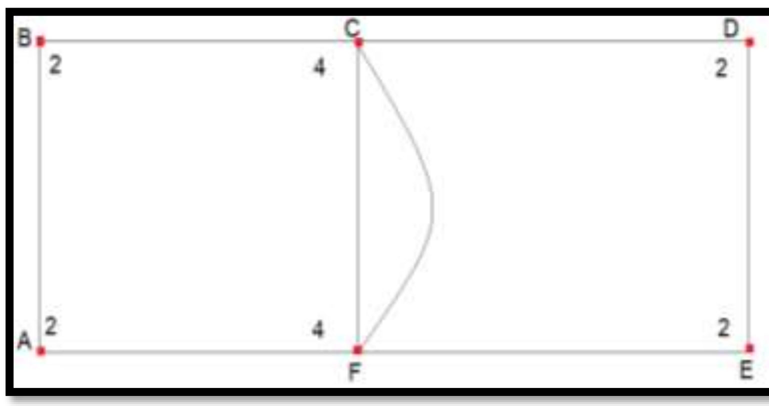
B.

This graph does not have an Euler path since all the vertices contains degree of 3 and it does not satisfy the theorem.

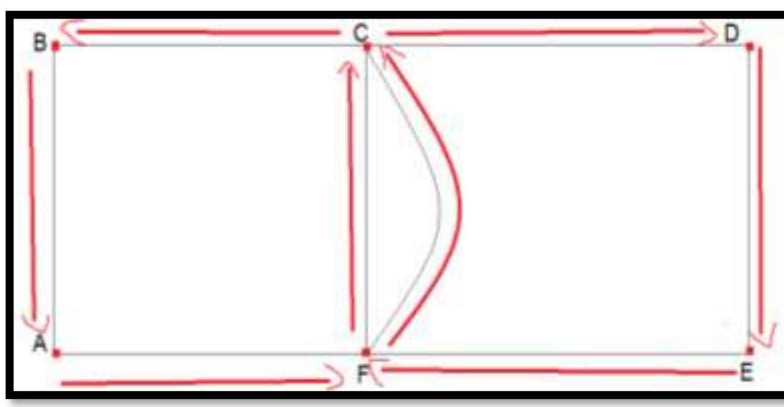


Example: Determine if each graph has a Euler circuit. If has one, find the Euler circuit.

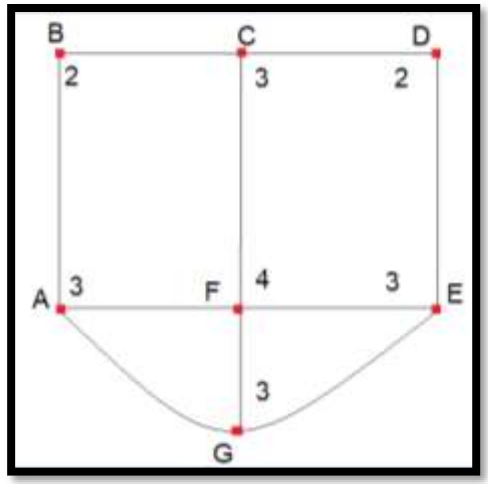
A. This graph has an Euler Circuit since it satisfy the Euler Circuit theorem that a graph will contain Euler circuit if all the vertices contains EVEN DEGREE.



The Euler Circuit is CBAFCDEFC



B. This graph does not have an Euler Circuit because it does not satisfy the Euler Circuit Theorem that a graph must contain all vertices a n EVEN DEGREE. And to this graph, the vertex A, C, E, G has an Odd degree.



For more knowledge about Euler Paths and Circuits please check the link provided;
https://study.com/academy/lesson/euler-paths-and-eulers-circuits.html

REMEMBER

- An Euler Circuit is always a Euler Path, but ...
  a Euler Path is not always a Euler Circuit.
- EULER'S PATH AND CIRCUIT THEOREMS
  - A graph will contain an Euler path if it contains at most two vertices of odd degree.
  - A graph will contain an Euler circuit if all vertices have even degree.

ACTIVITY:

Follow the steps with your imagination.

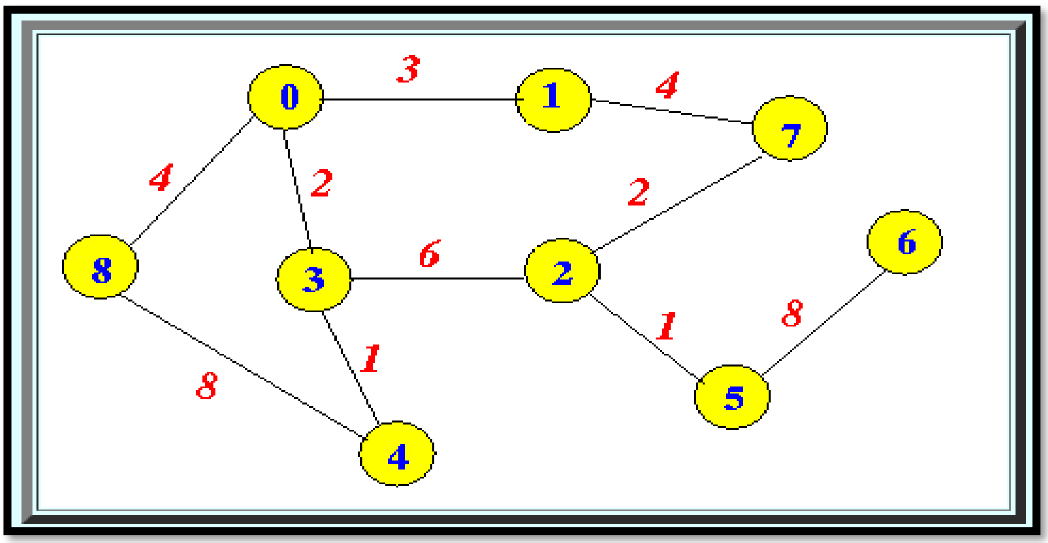
1. Take a trip through the Luneta Park.
2. Fly the friendly skies and pretend you are the pilot.
3. Navigate a trip to see all different city from your house to the destination.

Lesson 2: Weighted Graphs

Weighted Graph

- is a graph whose edges have weights.
  In many applications, each edge of a graph has an associated numerical value, called a weight. Usually, the edge weights are nonnegative integers. Weighted graphs may be either directed or undirected.

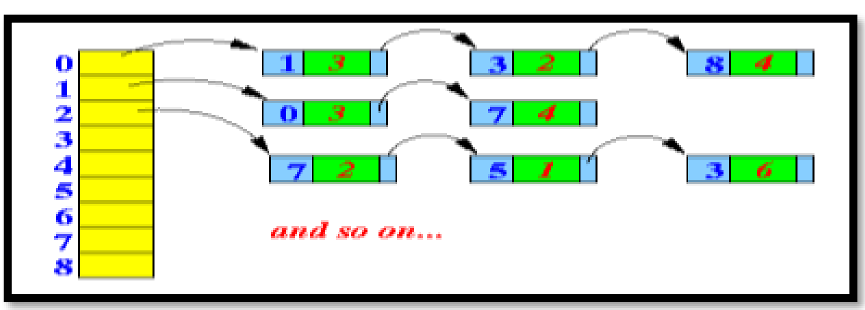
- The weight of an edge is often referred to as the "cost" of the edge. In applications, the weight may be a measure of the length of a route, the capacity of a line, the energy required to move between locations along a route, etc.



The weight of an edge can represent:

- **Cost or distance** = the amount of effort needed to travel from one place to another
- **Capacity** = the maximim amount of flow that can be transported from one place to another

Representation (for the graph above):



Hamiltonian Circuit and Hamiltonian Path

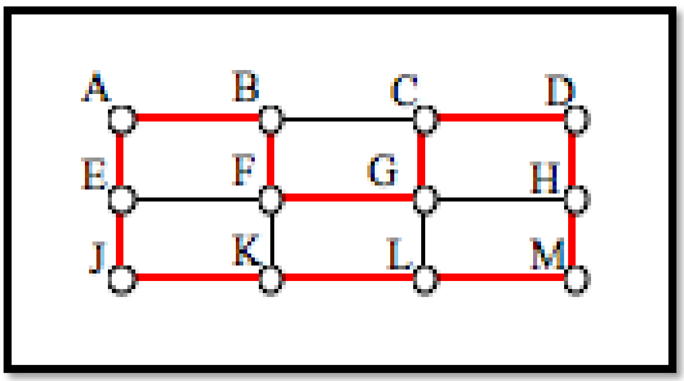
- A **Hamiltonian circuit** is a circuit that visits every vertex once with no repeats. Being a circuit, it must start and end at the same vertex.

- A **Hamiltonian path** also visits every vertex once with no repeats, but does not have to start and end at the same vertex.

Example:

One Hamiltonian circuit is shown on the graph below. There are several other Hamiltonian circuits possible on this graph. Notice that the circuit only has to visit every vertex once; it does not need to use every edge.

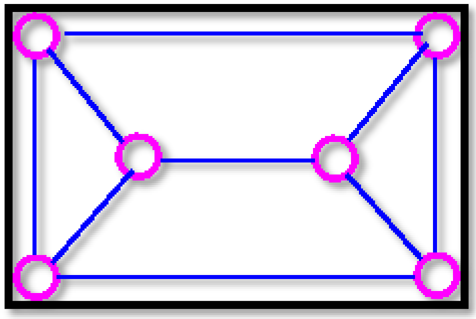
This circuit could be notated by the sequence of vertices visited, starting and ending at the same vertex: ABFGCDHMLKJEA. Notice that the same circuit could be written in reverse order, or starting and ending at a different vertex.



Dirac's Theorem

- Consider a connected graph with at least three vertices and no multiple edges. Let *n* be the number of vertices in the graph. If every vertex has degree of at least n/2, then the graph must be Hamiltonian.

Example:
Let G be a simple graph with n vertices where $n \geq 3$ If $\deg(v) \geq 1/2\ n$ for each vertex v, then G is Hamiltonian.

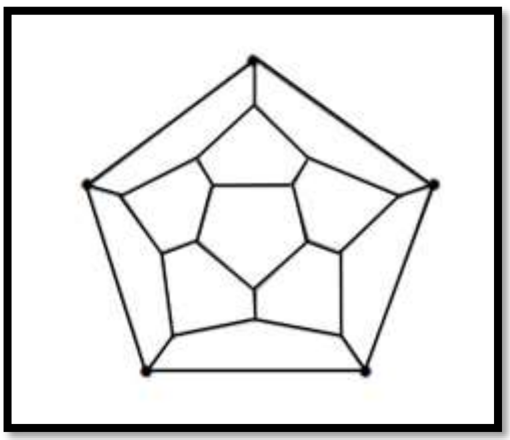


Solution:

n = 6 and deg(v) = 3 for each vertex, so this graph is Hamiltonian by Dirac's theorem.

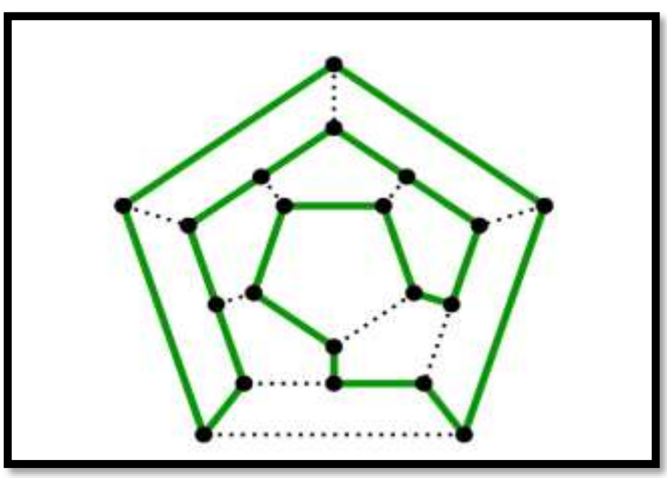
Example:

Does the following graph have a Hamiltonian Circuit?



Solution:

Yes, the above graph has a Hamiltonian circuit. The solution is…



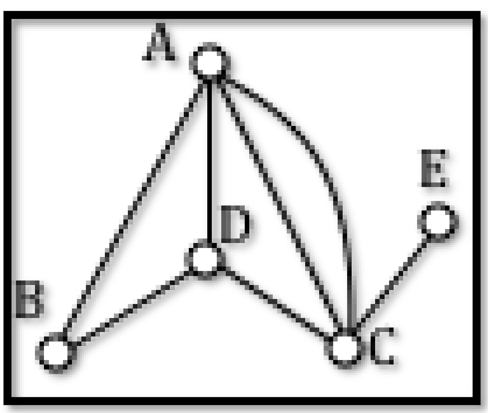
For more knowledge about Weighted Graph, please check the link provided;
https://www.britannica.com/science/Hamilton-circuit
https://courses.lumenlearning.com/math4liberalarts/chapter/introduction-euler-paths/

REMEMBER

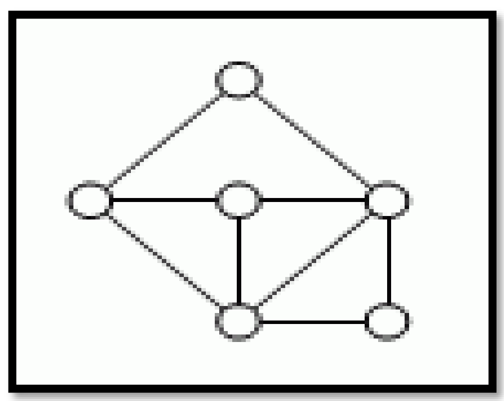
- Weighted Graph is a graph whose edges have weights.
- A **Hamiltonian circuit** is a circuit that visits every vertex once with no repeats. Being a circuit, it must start and end at the same vertex.
- A **Hamiltonian path** also visits every vertex once with no repeats, but does not have to start and end at the same vertex.
- Consider a connected graph with at least three vertices and no multiple edges. Let *n* be the number of vertices in the graph. If every vertex has degree of at least n/2, then the graph must be Hamiltonian.

APPLICATION

ACTIVITY:
Does a Hamiltonian path or circuit exist on the graph below? Explain briefly.



1st graph



2nd graph

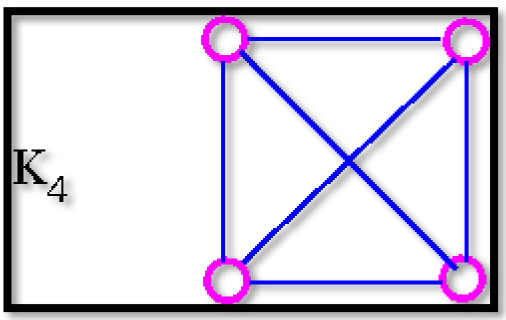
Lesson 3: Planarity and Euler's Formula

Planar Graph

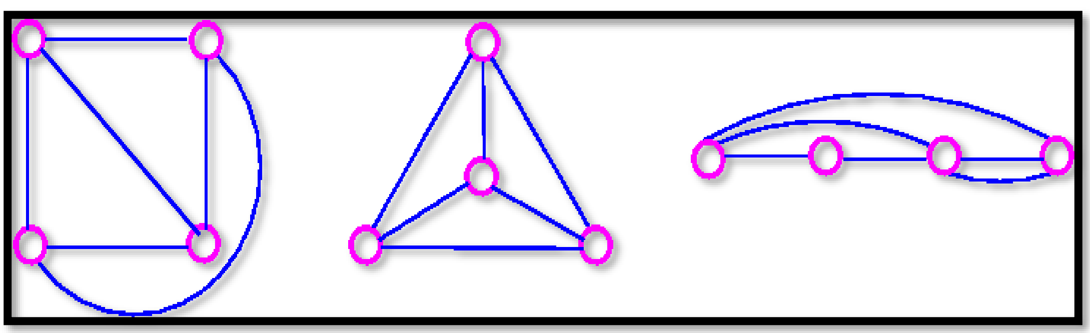
A graph G is planar if it can be drawn in the plane in such a way that no two edges meet each other except at a vertex to which they are incident. Any such drawing is called a plane drawing of G.

Example:

For example, the graph K_4 (4 vertices) is planar, since it can be drawn in the plane without edges crossing.

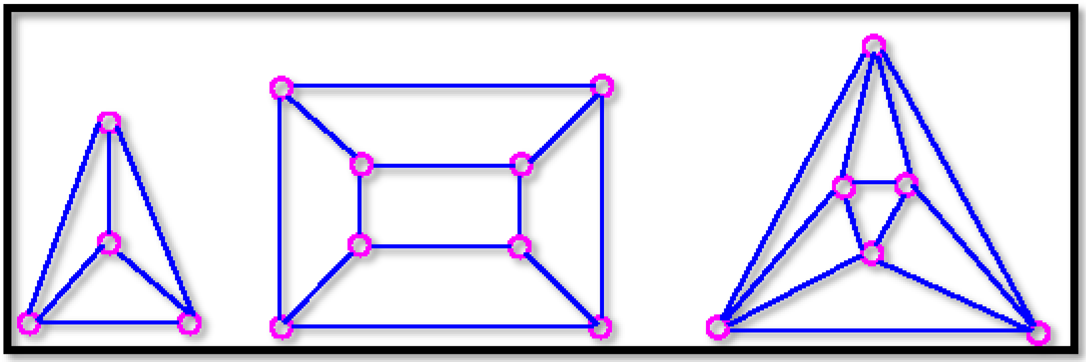


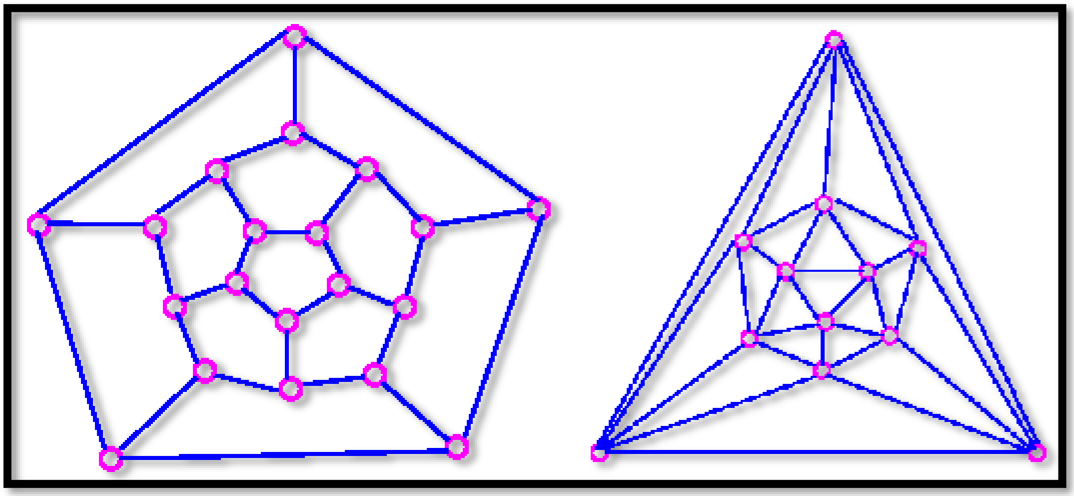
We can draw the K4 with three plane shown below:



Example:

The five Platonic graphs are all planar.

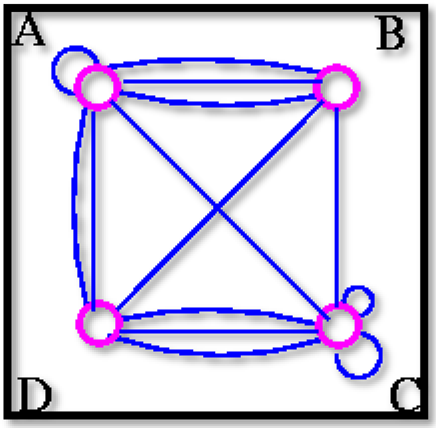


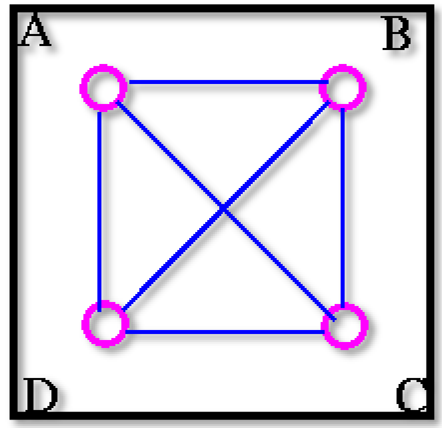


On the other hand, the complete bipartite graph $K_{3,3}$ is not planar, since every drawing of $K_{3,3}$ contains at least one crossing. why? because $K_{3,3}$ has a cycle which must appear in any plane drawing.

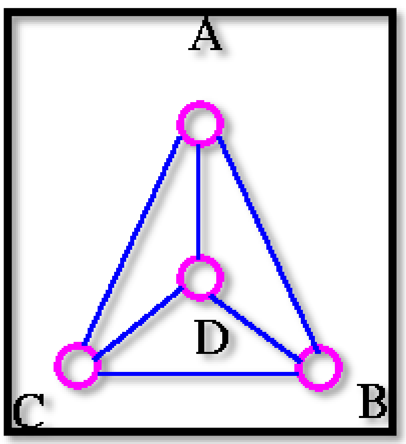
To study planar graphs, we restrict ourselves to simple graphs.

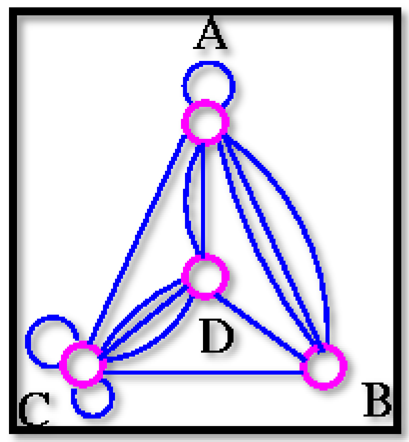
- If a planar graph has multiple edges or loops.
    - Collapse the multiple edges to a single edge.
    - Remove the loops.
- Draw the resulting simple graph without crossing.
- Insert the loops and multiple edges.





Remove loops and multiple edge.



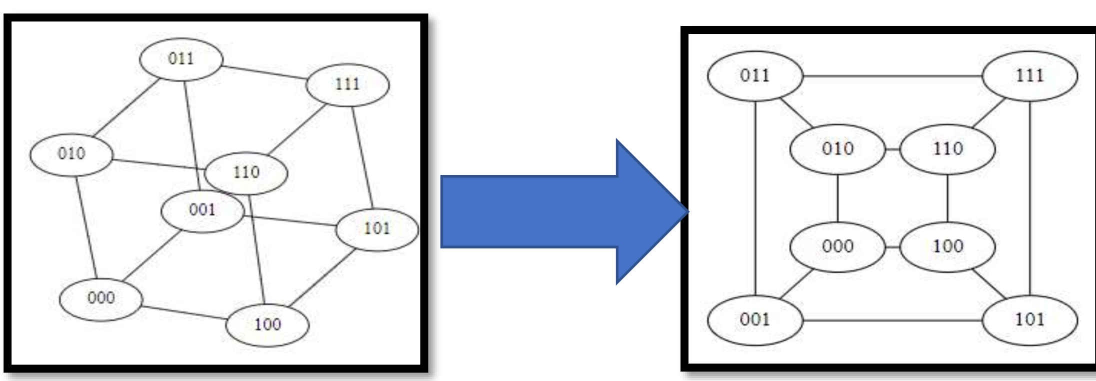


Draw without multiple edge.

Insert loops and multiple edges.

Example:

For example, we drew *Q*3 in a non-planar way originally, but it is actually planar:



Euler's Formula

- If G is a planar graph, then any plane drawing of G divides the plane into regions, called **faces.**

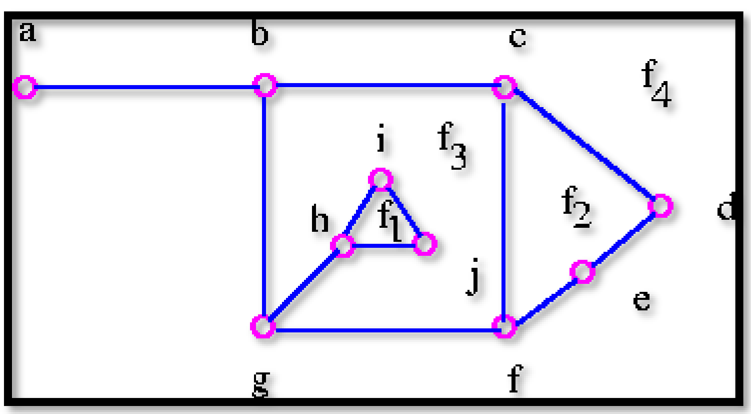
- One of these faces is unbounded, and is called the **infinite face.**

- If *f* is any face, then the degree of *f* (denoted by deg *f*) is the number of edges encountered in a walk around the boundary of the face *f*.

- If all faces have the same degree (g, say), the G is face-regular of degree g.

  For example:

  The following graph G has four faces, f_4 being the infinite face.



It is easy to see from above graph that deg f_1=3, deg f_2=4, deg f_3=9, deg f_4=8.

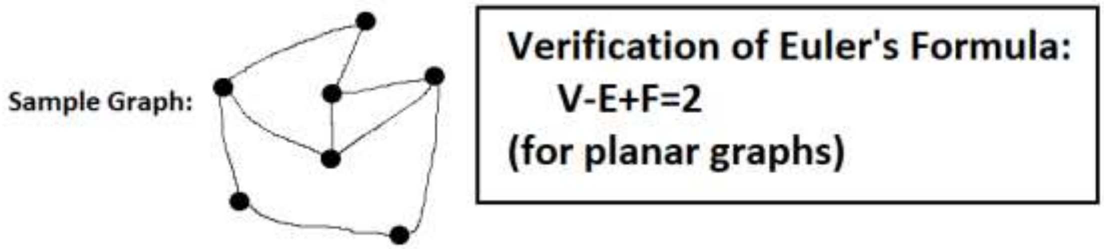
Note that the sum of all the degrees of the faces is equal to twice the number of edges in the graph , since each edge either borders two different faces (such as bg, cd, and cf) or occurs twice when walk around a single face (such as ab and gh).

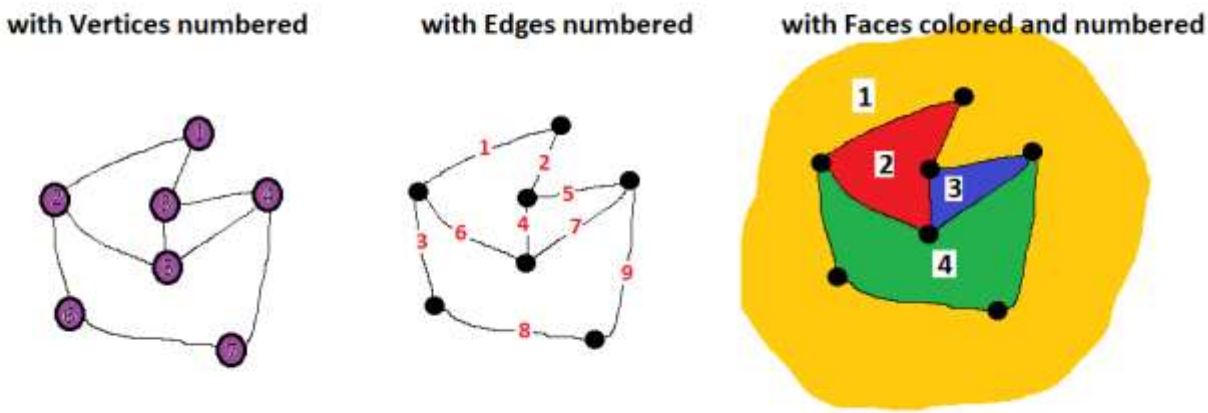
The Euler's formula relates the number of vertices, edges and faces of a planar graph. If *v*, *e*, and *f* denote the number of vertices, edges, and faces respectively of a connected planar graph, then we get ***$v-e+f = 2$*** or ***$v+f=e+2$.***

The Euler formula tells us that all plane drawings of a connected planar graph have the same number of faces namely, 2+*m*-*n*.

Theorem 1 (Euler's Formula): *Let G be a connected planar graph, and let **v**, **e** and **f** denote, respectively, the numbers of vertices, edges, and faces in a plane drawing of G. Then **v - e + f = 2**.*

Example:





Solution:

As we can see this figure (or "graph") has:

7 Vertices;

9 Edges;

4 Faces.

Euler's Formula says

XXX V−E+F=2

and for the sample (above) since

XXX 7−9+4=2

we have verified Euler's Formula (for this example).

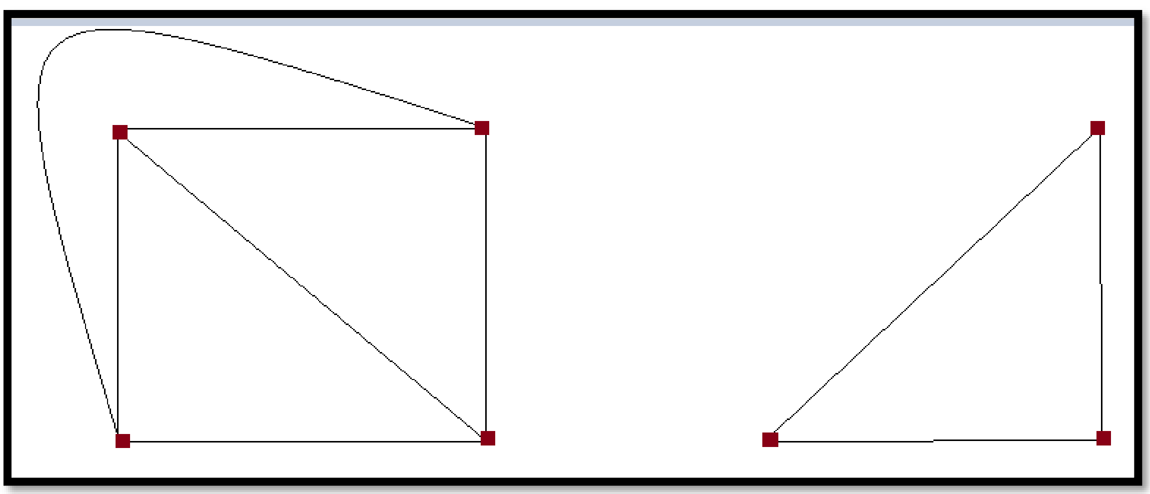
For more knowledge about Planarity and Euler's Formula, please check the link provided;
https://study.com/academy/lesson/euler-s-formula-for-planar-graphs.html
http://discrete.openmathbooks.org/more/mdm/sec_planar.html

REMEMBER

- A graph G is planar if it can be drawn in the plane in such a way that no two edges meet each other except at a vertex to which they are incident. Any such drawing is called a plane drawing of G.
- In a connected planar graph drawn with no intersecting edges, let *v* be the number of vertices, *e* be the number of edges, and *f* be the number of faces. Then $\boldsymbol{v - e + f = 2}$.

ACTIVITY:

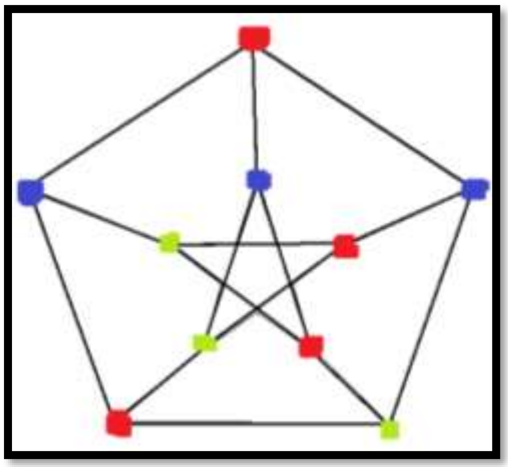
1. Show that the first graph below can be constructed for the second graph.
2. Verify the Euler's formula in the first graph.



Lesson 4: Graph Coloring

Coloring of Graph

A proper coloring of a graph is an assignment of colors to the vertices of the graph so that no two adjacent vertices have the same color.

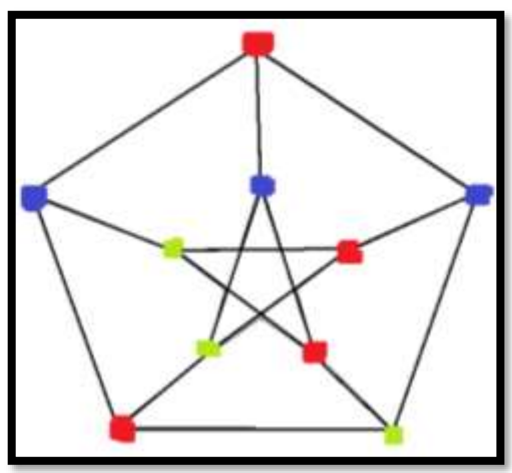


Chromatic Number

The chromatic number of a graph is the least number of colors need to make a colouring.

Example: This diagram shows the minimum coloring of the "Peterson" graph. What is the chromatic number?

3



Steps in Coloring a Graph

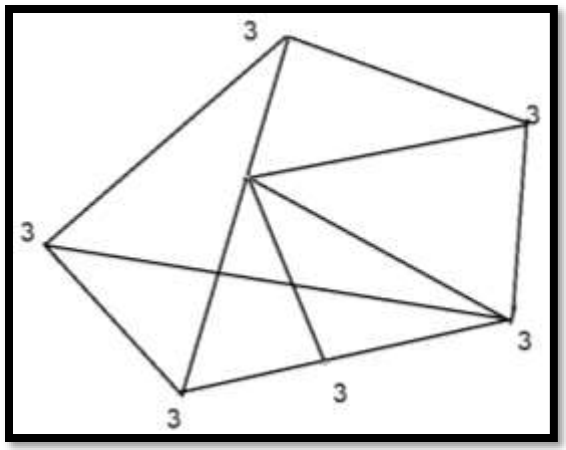
Step 1: Choose a vertex with highest degree, and color it. Use the same color as many vertices as you can without coloring vertices joined by an edge of the same color.

Step 2: Choose a new color, and repeat what you did in Step 1 for vertices not already colored.

Step 3: Repeat Step 1 until all vertices are colored.

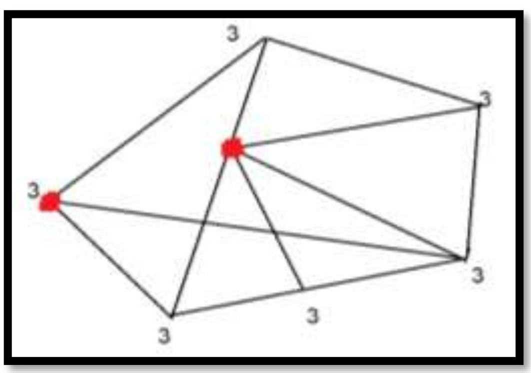
Example:

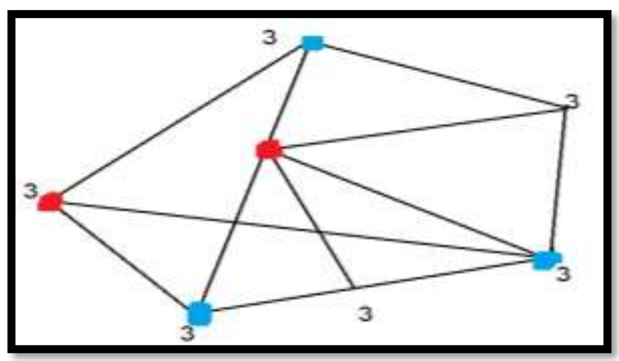
Color the graph below and give its chromatic number.

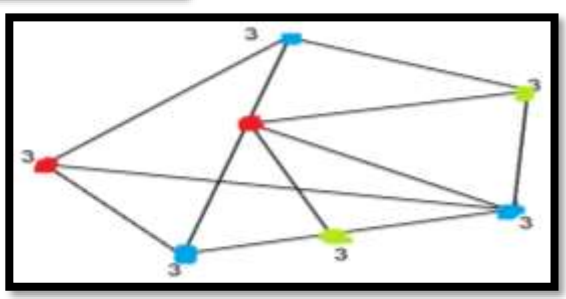


Solution:

Start with the highest degree and color it. Use the same color to color as many vertices as you can without coloring vertices joined by an edge of the same color. Repeat the step until you are done. So for that graph, the chromatic number is three.





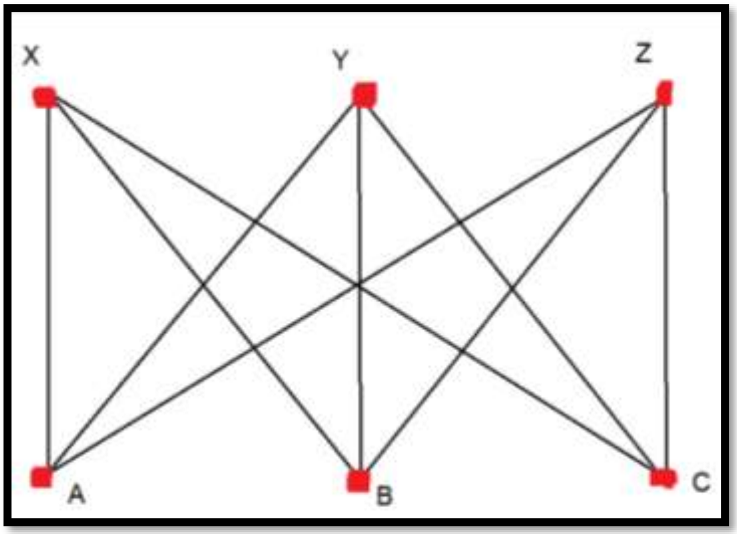


2-colorable Graph Theorem

A graph is 2-colorable if and only if it has no circuits that consist of an odd number of vertices.

Example:

Find the chromatic number of the graph



Solution:

Note that the graph contains circuits such as A-Y-C-Z-B-X-A with six vertices and A-Y-B-X-A with four vertices. It seems that any circuit we find in fact, involves an even number of vertices. It is difficult to determine whether we have looked at all possible circuits, but our observations suggest that the graph may be 2-colorable. A little trial and error confirm this if we simply color vertices A, B and C one color and the remaining vertices another. Thus, this graph has a chromatic number of 2.

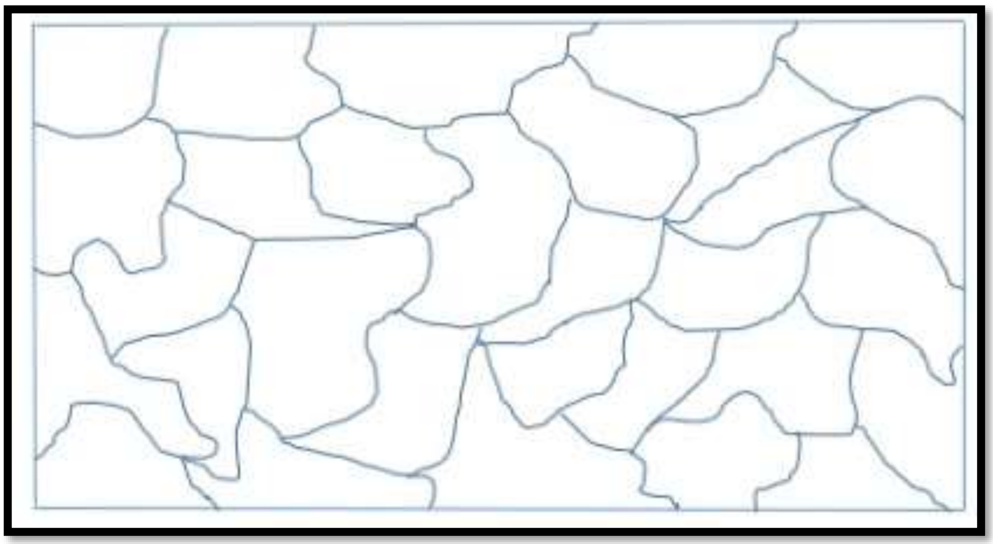
Map Coloring

Four Color Theorem

Every planar graph is 4-colorable. (In some case less than four colors may be required. Also if the graph is planar, more than four colors may be necessary.)

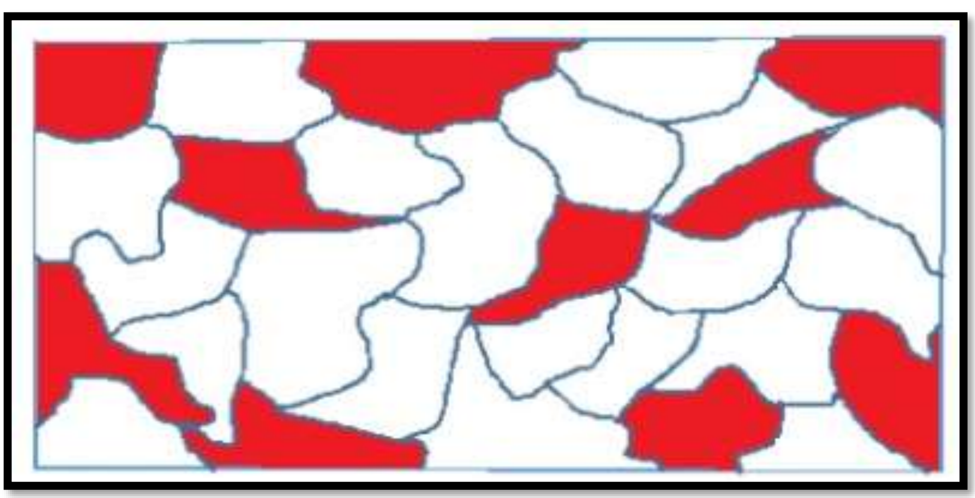
Example:

First let us take a planar graph of 30 regions and try to color it with the help of Four-Color Theorem.

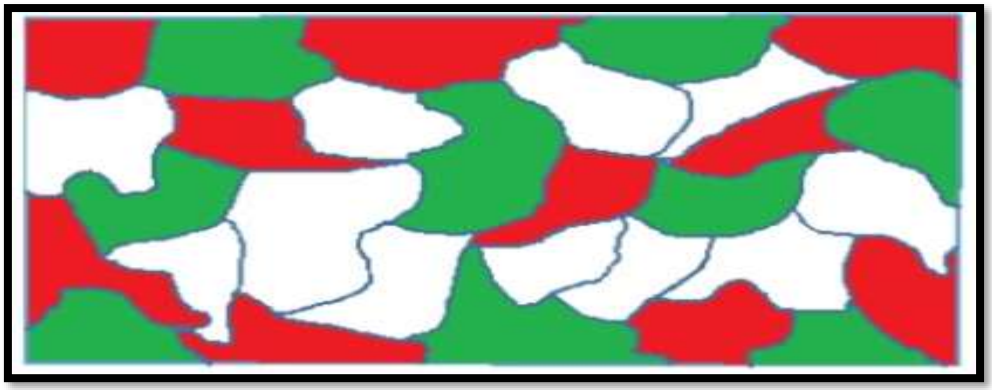


Solution:

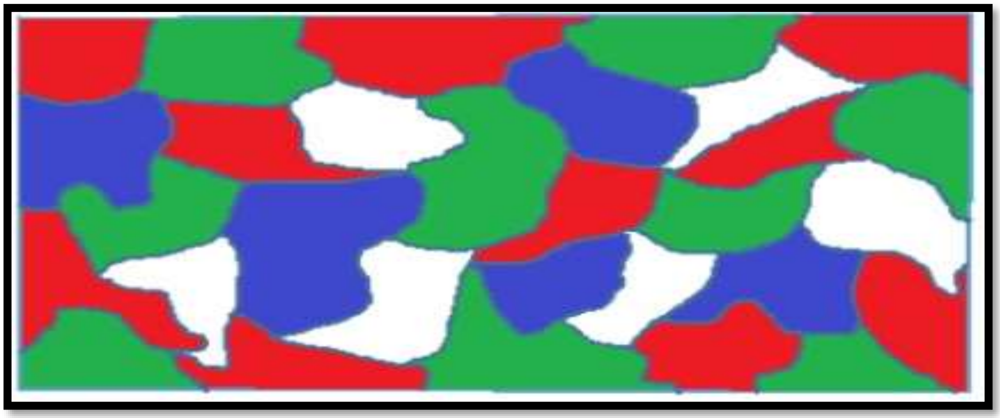
Now we want to color the graph with the first color red.



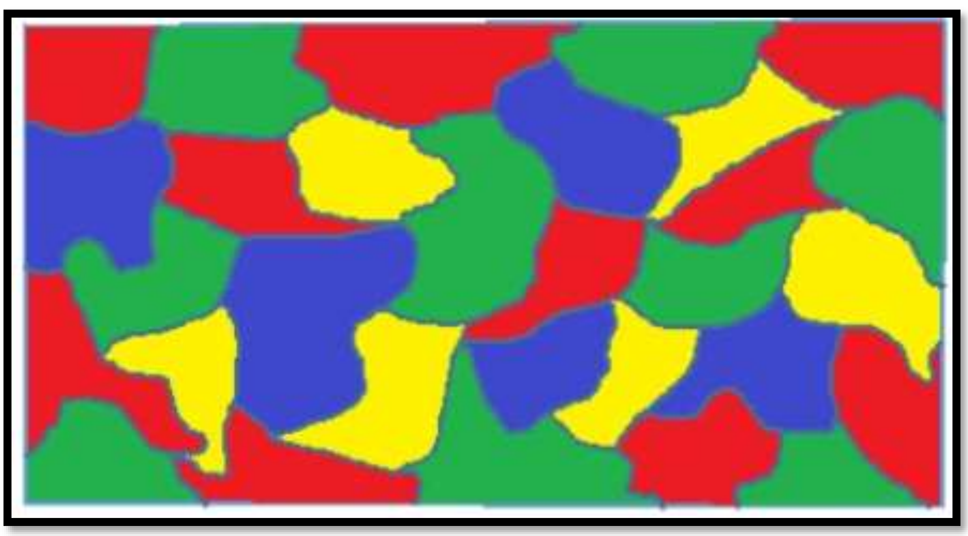
Next is to color it with green.



Next with the color blue.



And last, color the last region with yellow to get the desired color of map with the use of four colors. Thus, this example given uses a four-color theorem.



For more knowledge about Graph coloring, please check the link provided;
http://discrete.openmathbooks.org/dmoi2/sec_coloring.html
https://www.slideshare.net/MdShahAlam23/map-coloring-and-some-of-its-applications

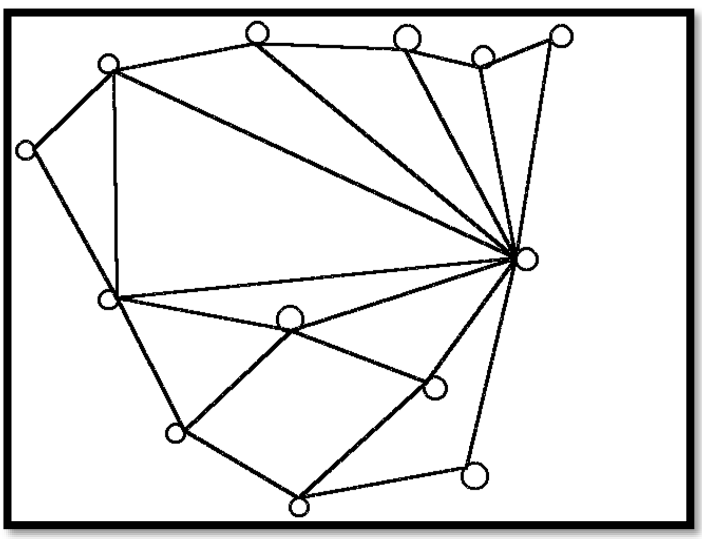
REMEMBER

- A proper coloring of a graph is an assignment of colors to the vertices of the graph so that no two adjacent vertices have the same color.
- **2-colorable theorem**
  - A graph is 2-colorable if and only if it has no circuits that consist of an odd number of vertices.
- **Four-colorable theorem**
  - Every planar graph is 4-colorable.

APPLICATION

ACTIVITY:

1. What is the chromatic number of this graph?
2. How many colors did you use to color the original map?



REFERENCES

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