

CHAPTER 9: Introduction to Mathematics of Money

**Objectives:**

- a. Differentiate simple interest and compound interest
- b. Find the future value, present value of ordinary annuity.
- c. Compute for simple interest, compound interest using the other formulas.

Lesson 1: Simple Interest**Simple Interest**

Is used on short-term notes- often on duration less than 1 year. The concept of simple interest, however, forms the basis of much of the rest of the lessons in the next two sections, for which time periods may be much longer than a year.

The Simple Interest I is given by:

$$I = Prt$$

Where: I = interest

P = principal

r = annual interest rate (written as a decimal)

t = time (in years)

- **Principal**
 - If we deposit a sum of money P in a savings account or if we borrow a sum of money P from a lender, then P is referred to as principal.
- **Interest**
 - When money is borrowed- whether it is a savings institution borrowing from us when we deposit money to our account, or we, borrowing from a lender – a fee is charge for the money borrowed. This fee is rent paid for the use of another’s money, just as rent is paid for the use of another’s house. The fee is called the interest (I).
- **Interest rate**
 - It is usually computed as a percentage called the *interest rate* (r) of the principal over a given period of time. The interest rate, unless otherwise stated, is an *annual rate*.

- **Length of time (t)**
 - Is the period that money was deposited or lent.

Example:

A bank pays simple interest at the rate of 8% per year for a certain deposit. If a customer deposits Php 10, 000 and make no withdrawals for three years, what is the interest earned in that period of time?

Solution:

Using the formula with the given:

$$P = \text{Php } 10,000 \qquad r = 0.08 \text{ (8\% converted in decimal number)} \qquad t = 3$$

$$I = Prt$$

$$I = 10,000(0.08)(3)$$

$$I = 2400$$

Php 2, 400 is the e=interest earned over the 3-year period.

Simple Accumulated Value

- **Accumulated Value or Accumulated Amount (A)**
 - The amount of money originally invested added to the total amount of interest earned on that investment.
- **Maturity Value (A) of a loan**
 - The total payback of principal and interest.

Once the interest have been calculated, the accumulated amount or maturity value A , which is the sum of the principal and interest after t years, is given by:

$$A = P + I = Prt \quad \text{or} \quad A = P(1 + rt)$$

Future Amount, Accumulated Amount or Maturity Value

$$A = P(1 + rt)$$

Where: A = accumulated amount, or future value, or maturity value

P = principal or present value

r = annual simple interest rate (written as a decimal)

t = time in years

Example:

An amount of Php 500,000 is invested in a 10-year trust that pays 6% annual simple interest. What is the total amount of the trust fund at the end of ten years?

Solution:

Using the formula

$$A = P(1 + rt)$$

the given are: $P = 500,000$ $r = 0.06$ (6% converted in decimal) $t = 10$

$$A = P(1 + rt)$$

$$A = 500,000[1 + 0.06(10)]$$

$$A = 800,000$$

Therefore, Php 800,00 is the total amount of trust fund at the end of ten years.

Time in months	Time in days (exact method)	Time in days (ordinary method)
$t = \frac{\text{number of months}}{12}$	$t = \frac{\text{number of days}}{365}$	$t = \frac{\text{number of days}}{360}$

In the simple interest formula, the interest rate is an annual rate. Therefore, when the time period of a loan is given in days or months, the time of the loan must be converted to years. To convert units of time from days or months to years. We use the formula above.

Note that when a loan is in terms of days, the time may be calculated by using either the exact method or ordinary method. The **ordinary method** is sometimes referred to as the **banker's method**, because this method is used by many banks.

Example:

Find the simple interest on a 45-day loan of Php 2,500 at an annual interest rate of 10.5% using the

- Exact method
- Ordinary method

Solution:

Use the simple interest formula with the given

$$P = 2,500 \qquad r = 0.105 \text{ (10.5\% converted in decimal)} \qquad I = Prt$$

$$\text{a. } t = \frac{45}{365} \qquad I = Prt = I = 2,500 (0.105) \left(\frac{45}{365}\right) \approx 32.36$$

The exact method yields interest of Php 32.36.

$$\text{b. } t = \frac{45}{360} \qquad I = Prt = I = 2,500 (0.105) \left(\frac{45}{360}\right) \approx 32.81$$

The ordinary method yields interest of Php 32.81. Note that the interest is greater when the ordinary method is used.

Example:

A loan of Php 25,000 is made for 9 months at a simple interest rate of 10% per annum. What is the interest charge? What amount is due after 9 months?

Solution:

The actual period for which money is borrowed is 9 months, which is $\frac{9}{12} = \frac{3}{4}$ of the year. The interest charge is the product of the amount borrowed, Php 25,000.; the annual rate of interest, 10% expressed as a decimal, 0.10; and the length of time in years, $\frac{3}{4}$. Use the Simple Interest Formula to find the interest charge I .

$$I = Prt$$

$$\text{Interest charge} = (25,000)(0.10)\left(\frac{3}{4}\right) = \text{Php } 1,875.$$

Using Future Value Formula, the amount A due after 9 months is

$$A = P + I$$

$$A = \text{Php } 25,000 + \text{Php } 1,875$$

$$A = \text{Php } 26,875$$



For more knowledge about Simple Interest, please check the link provided;
<https://study.com/academy/lesson/how-to-find-simple-interest-rate-definition-formula-examples.html>

REMEMBER



- The Simple Interest I is given by:

$$I = Prt$$
- Future Amount, Accumulated Amount or Maturity Value

$$A = P + I = Prt \quad \text{or} \quad A = P(1 + rt)$$



APPLICATION

ACTIVITY: Solve the given problem and show your complete solution.

Jessa borrows Php 1, 000, 000 for 1 month at a simple interest rate of 9% per annum. How much must Jessa pay back at the end of 1 month?

Lesson 2: Compound Interest and Continuous Compound Interest

Compound Interest

- A second method of paying interest where the interest for each period is added to the principal before interest is calculated for the next period.
- With compound interest, both the interest added and the principal each interest for the next period.
- With this method, the principal grows as the interest is added to it.
- This method is used in investments such as savings accounts and bonds

In working with problems involving interest, we use the term payment period as shown below:

Annually	Once per year
Semiannually	Twice per year
Quarterly	4 times a year (every 3 months)
Monthly	12 times per year
Daily	365 times per year

If the interest due at the end of each payment period is added to the principal, so that the interest computed for the next payment period is based on this new amount of the old principal plus interest, then the interest is said to have been compounded.

That is, **compound interest** is interest paid on the initial principal and previously earned interest.

Example:

Suppose you deposit Php 10,000 in a bank that pays 8% compounded quarterly. How much will the bank owe you at the end of the year?

- **Compounded Quarterly** means that the earned interest is paid to your account at the end of each 3-month period and that interest as well as the principal earns interest for the next quarter.

Solution:

Using the simple interest formula from the previous lesson, we compute the amount in the account at the end of the first quarter after interest has been paid:

$$A_1 = P(1 + rt)$$

$$A_1 = \text{Php } 10,000 \left[1 + 0.08 \left(\frac{1}{4} \right) \right]$$

$$A_1 = \text{Php } 10,000 (1.02)$$

$$A_1 = \text{Php } 10,200$$

Now, Php 10,200 is the new principal for the second quarter. At the end of the second quarter, after interest is paid, the account will have:

$$A_2 = P(1 + rt) = A_1 (1 + rt)$$

$$A_2 = \text{Php } 10,000 \left[1 + 0.08 \left(\frac{1}{4} \right) \right] \left[1 + 0.08 \left(\frac{1}{4} \right) \right]$$

$$A_2 = \text{Php } 10,000 \left[1 + 0.08 \left(\frac{1}{4} \right) \right]^2$$

$$A_2 = \text{Php } 10,000 (1.02)^2$$

$$A_2 = \text{Php } 10,404$$

Similarly, at the end of the third quarter you will have:

$$A_3 = P(1 + rt) = A_2 (1 + rt)$$

$$A_3 = \text{Php } 10,000 \left[1 + 0.08 \left(\frac{1}{4} \right) \right]^2 \left[1 + 0.08 \left(\frac{1}{4} \right) \right]$$

$$A_3 = \text{Php } 10,000 (1.02)^3$$

$$A_3 = \text{Php } 10,612.08$$

Finally, the accumulated amount at the end of the fourth quarter is

$$A_4 = P(1 + rt) = A_3 (1 + rt)$$

$$A_4 = \text{Php } 10,000 \left[1 + 0.08 \left(\frac{1}{4} \right) \right]^3 \left[1 + 0.08 \left(\frac{1}{4} \right) \right]$$

$$A_4 = \text{Php } 10,000 (1.02)^4$$

$$A_4 = \text{Php } 10,824.32$$

Compound interest vs. Simple Interest

If we used the amount for simple interest we got:

$$A = P(1 + rt)$$

$$A = \text{Php } 10,000[1 + 0.08(1)]$$

$$A = \text{Php } 10,000(1.08)$$

$$A = \text{Php } 10,800$$

We see that compounding quarterly yields Php 24.32 more than simple interest would provide.

In general, if P is the principal earning interest compounded m times a year at an annual rate of j , then (by repeated use of the simple interest formula, using $i = \frac{j}{m}$, the rate per period) the amount A at the end of each period is:

$A_1 = P(1 + i)$	End of first period
$A_2 = [P(1 + i)](1 + i)$ $A_2 = P(1 + i)^2$	End of second period
$A_3 = [P(1 + i)^2](1 + i)$ $A_3 = P(1 + i)^3$	End of third period
$A_n = [P(1 + i)^{n-1}](1 + i)$ $A_n = P(1 + i)^n$	End of nth period

Compound Interest Formula

$$A = P(1 + i)^n \quad (1)$$

Where: $i = \frac{j}{m}$, $n = mt$, and

A = accumulated amount at the end of n conversion periods

P = principal

j = nominal interest rate per year (annual rate (r))

m = number of conversion periods per year

i = rate per conversion period (interest rate per year)

n = total number of conversion periods

t = term (number of years)

The formula below can be derived in much the same way as the Compound Interest Formula (1).

Compound Interest Formula (1) is equivalent to the formula below

$$A = P \left[1 + \frac{j}{m} \right]^{mt} \quad (2)$$

Where: t is the time, in years, that the principal is invested.

For a compound interest calculation, formula (2) may seem more natural to use than (1), if r (the natural interest rate) and t (time in year) are given. On the other hand, if i (the interest rate per period), and n (the number of compounding periods) are given, formula (1) may seem easier to use.

Example:

For each of the following investments, determine the interest rate per period i , and the number of compounding periods n .

- 12% compounded monthly for 7 years
- 7.2% compounded quarterly for 11 years
- 10% compounded daily for 5 years

Solution:

- If the compounding is monthly and $j = 12\% = 0.12$, then

$$i = \frac{j}{m} \qquad i = \frac{0.12}{12} = 0.01$$

The number of compounding periods is

$$n = (7 \text{ years}) \left(\frac{12 \text{ period}}{\text{year}} \right)$$

$$n = 84$$

- If the compounding is quarterly and $j = 7.2\% = 0.072$, then

$$i = \frac{j}{m} \qquad i = \frac{0.072}{4} = 0.018$$

The number of compounding periods is

$$n = (11 \text{ years}) \left(\frac{4 \text{ quarter}}{\text{year}} \right)$$

$$n = 44$$

- c. If the compounding is daily and $j = 10\% = 0.10$, then

$$i = \frac{j}{m} \qquad i = \frac{0.10}{365} \approx 0.00027$$

The number of compounding periods is

$$n = (5 \text{ years}) \left(\frac{365 \text{ days}}{\text{year}} \right)$$

$$n = 1,825$$

Continuous Compound Interest

More frequent compounding means that interest is paid more often (and hence more interest is earned), it would seem that the more frequently the interest is compounded, the larger future value will become.

Using a calculator with a y^x key and the Compound Interest Formula (2) gives a chart table that shows the future value that results as the number of compounding periods increase, given the following conditions:

Interest on Php 1,000 at 12% per year for 10 years

Compounded	Number of periods of Compounding	Future value or Compound Amount	Interest
Not at all (simple interest)	---	---	1,200
Annually	$1 \times 10 = 10$	$1,000 \left(1 + \frac{0.12}{1} \right)^{10} \approx 3,105.85$	2,105.85
Semiannually	$2 \times 10 = 20$	$1,000 \left(1 + \frac{0.12}{2} \right)^{20} \approx 3,207.14$	2,207.14
Quarterly	$4 \times 10 = 40$	$1,000 \left(1 + \frac{0.12}{4} \right)^{40} \approx 3,262.04$	2,262.04
Monthly	$12 \times 10 = 120$	$1,000 \left(1 + \frac{0.12}{12} \right)^{120} \approx 3,300.39$	2,300.39
Daily	$365 \times 10 = 3,650$	$1,000 \left(1 + \frac{0.12}{365} \right)^{3650} \approx 3,319.46$	2,319.46

Hourly	$24 \times 365 \times 10 = 87,600$	$1,000 \left(1 + \frac{0.12}{8,760}\right)^{87,600} \approx 3,320.09$	2,320.09
Each minute	$60 \times 24 \times 365 \times 10$ $= 5,256,000$	$1,000 \left(1 + \frac{0.12}{5,256,000}\right)^{5,256,000} \approx 3,320.12$	2,320.12

Observe that the accumulated mounts corresponding to interest compounded daily and the interest compounded continuously differ by very little. This is true even for a process called **continuous compounding**, which can be loosely described as compounding every instant.

Continuous Compound Interest Formula

If a principal P is invested at an annual rate i (expressed as a decimal) compounded continuously, then the amount A in the account at the end of n years is given by:

$$A = Pe^{mt} \quad (3)$$

The number e comes up in various practical application (to five decimal places, $e = 2.71828 \dots$)

Example:

Suppose Php 5,000 is deposited in an account paying 8% compounding continuously for five years. Find the compounded amount.

Solution:

$$\text{Let } P = 5,000 \quad n = 5 \quad i = 0.08$$

$$\text{Then, } A = 5,000e^{5(0.08)} = 5,000e^{0.4}$$

since using a calculator, we know that $e^{0.4} \approx 1.49182$ and so

$$A = 5,000(1.49182) = 7,459.20 \quad \text{or Php } 7,459.10$$

Effective Rate of Interest

One way of comparing interest rates is provided by the use of the *effective rate*. The effective rate is the simple interest rate that would produce the same accumulated amount in 1 year as the nominal rate compounded m times a year. The effective rate is also called the **effective annual yield**.

To derive a relationship between nominal interest rate i per year compounded m times, and its corresponding effective rate R per year, let us assume an initial investment of P pesos. Then the accumulated amount after 1 year at a simple interest rate of R per year is

$$A = P(1 + R)$$

Since $i = \frac{j}{m}$, the accumulated amount after 1 year at an interest rate j per year compounded m times a year is

$$A = P(1 + i)^n = P \left[1 + \frac{j}{m} \right]^m$$

Equating the two expressions gives

$$P(1 + R) = P \left[1 + \frac{j}{m} \right]^m$$

$$(1 + R) = \left[1 + \frac{j}{m} \right]^m$$

Or, upon solving for R , we obtain the following formula for computing the effective rate of interest.

Effective Rate of Interest Formula

$$R = \left[1 + \frac{j}{m} \right]^m - 1 \quad (4)$$

Where

R = effective rate of interest

j = nominal interest rate per year

m = number of conversion periods per year

Example:

Find the effective rate of interest corresponding to a nominal rate of 8% per year compounded (a) annually, (b) semiannually, (c) quarterly, (d) monthly, and (e) daily.

Solution:

- a. The effective rate of interest corresponding to nominal rate of 8% per year compounded annually is of course given by 8% per year. This result is also confirmed by using Formula (4) with $i = 0.08$ and $m = 1$. Thus,

$$R = (1 + 0.08) - 1$$

So, the corresponding effective rate in this case is 8% per year.

- b. Let $i = 0.08$ and $m = 2$. Then, formula (4) yields

$$R = \left[1 + \frac{0.08}{2}\right]^2 - 1$$
$$R = [1 + 1.04]^2 - 1$$
$$R = 0.0816$$

So, the corresponding effective rate in this case is 8.16% per year.

- c. Let $i = 0.08$ and $m = 4$. Then, formula (4) yields

$$R = \left[1 + \frac{0.08}{4}\right]^4 - 1$$
$$R = [1 + 1.02]^2 - 1$$
$$R \approx 0.08243$$

So, the corresponding effective rate in this case is 8.24% per year.

- d. Let $i = 0.08$ and $m = 12$. Then, formula (4) yields

$$R = \left[1 + \frac{0.08}{12}\right]^{12} - 1$$
$$R \approx 0.08300$$

So, the corresponding effective rate in this case is 8.300% per year.

- e. Let $i = 0.08$ and $m = 365$. Then, formula (4) yields

$$R = \left[1 + \frac{0.08}{365}\right]^{365} - 1$$
$$R \approx 0.08328$$

So, the corresponding effective rate in this case is 8.328% per year.



For more knowledge about Compound Interest, please check the link provided;

<https://www.thecalculatorsite.com/articles/finance/compound-interest-formula.php>

REMEMBER



- **Compound Interest Formula**

$$A = P(1 + i)^n \quad (1)$$

$$A = P \left[1 + \frac{j}{m} \right]^{mt} \quad (2)$$

- **Continuous Compound Interest Formula**

$$A = Pe^{mt} \quad (3)$$

- **Effective Rate of Interest Formula**

$$R = \left[1 + \frac{j}{m} \right]^m - 1$$



APPLICATION

ACTIVITY: Solve the given problem and show your complete solution.

What amount must be invested now in order to have Php 1,200,000 after 3 years if money is worth 6% compounded semiannually?

Lesson 3: Ordinary Annuity

Annuity

- An annuity is a contract between you and an insurance company in which you make a lump-sum payment or series of payments and, in return, receive regular disbursements, beginning either immediately or at some point in the future.
- Some examples of annuity are:
 - a. Regular deposits to a savings account
 - b. Monthly payments of house rent
 - c. Monthly retirement benefits from a pension loan
 - d. Installment payments in purchasing a condominium

Term of the Annuity

- The time period in which these payments are made.
- The terms of the annuity may be annually, semiannually, quarterly, or at other intervals.

Classification of Annuities by correspondence with interest periods:

- **Simple Annuity**

- Is an annuity where the payment interval coincides with the interest conversion period.

Example:

Php 10,000 is invested at the end of every six month for 2 years and that 15% interest is paid compounded semiannually. Here the term is 2 years with the periodic payment of Php 10,000 every 6 months, that is, the payment period is equal to the interest period semiannually.

Three Kinds of Simple Annuity

- Annuity Certain – the term is given by a fixed time interval
- Perpetuity – a time interval that begins at a definite date but extends indefinitely
- Contingent Annuity – one that is not fixed in advance

- **General Annuity**

- Is annuity where the payment interval does not coincide with the interest conversion period.

Example:

Every 3 months, for 8 years, a father deposits Php 15,000 in a trust fund for his son's education. The money earns at 12% compounded monthly. The payment period is not equal to the interest period.

Ordinary Annuity (annuity immediate)

- Classification of annuity by payment dues. This is an annuity in which the payments are made at the *end* of each payment period.

Annuity Due

- An annuity in which the payments are made at the beginning of each period.

Complex Annuity

- An annuity in which the payment period does not coincide with the interest conversion period.

Ordinary Annuities are considered as certain and simple with periodic payments that are equal in size. In other words, we study annuities that are subject to the following conditions:

1. The terms are given by fixed time intervals.
2. The periodic payments are equal in size.
3. The payments are made at the *end* of the payment periods.
4. The payment periods coincide with the interest conversion periods.

The sum of all payments plus all interest earned is called the *future amount of the annuity* or *maturity value* or its *future value*.

Future Value of an Ordinary Annuity

Future Value of an Ordinary Annuity

If Php R is deposited at the end of each period for n periods in an annuity that earns interest at a rate of i period, the future value of the annuity will be

$$S = R \cdot s_{n|i} = R \left[\frac{(1+i)^n - 1}{i} \right]$$

Where $s_{n|i}$ is read “s, n angle i ” and represents the future value of an ordinary annuity of Php 1 per period for n periods with an interest rate of i per period.

Example:

Mr. Rosales deposits Php 75,000 at the end of each month in a bank which pays 9% compounded monthly. If no withdrawals are made, how much could he expect to have after 4 years?

Solution:

We are looking for the amount S given that

$$R = \text{Php } 75,000 \quad m = 12 \quad j = 0.09 \quad t = 4$$

Get the value of i and n by

$$i = \frac{j}{m} = \frac{0.09}{12} = 0.0075 \quad n = t \cdot m = (4)(12) = 48$$

So we have now

$$S = 75000 \cdot s_{48|0.0075} = 75000 \left[\frac{(1 + 0.0075)^{48} - 1}{0.0075} \right]$$

$$S \approx 4,314,053.33$$

Present Value of an Ordinary Annuity

Present Value of an Ordinary Annuity

If a payment of Php R is to be made at the end of each period for n periods from an account that earns interest at a rate of i per period, then the account is an ordinary annuity, and the present value is given by:

$$A_n = R \cdot a_{n|i} = R \left[\frac{1 - (1 + i)^{-n}}{i} \right]$$

Where $a_{n|i}$ represents the present value of an ordinary annuity of Php 1 per period for n periods, with an interest rate of i per period.

The value of $a_{n|i}$, can also be computed directly with a calculator.

Example:

What is the present value of an annuity of Php 150,000 payable at the end of each 6-month period for 2 years if money is worth 8%, compounded semiannually?

Solution:

It is given that

$R = \text{Php } 150,000$ and $i = \frac{j}{m} = \frac{0.08}{2} = 0.04$. Because a payment is made twice a year for 2 years, the number of periods is $n = t \cdot m = (2)(2) = 4$. Thus,

$$\begin{aligned} A_n &= 150,000 \cdot a_{4|0.04} \\ &= 150,000 \left[\frac{1 - (1 + 0.04)^{-4}}{0.04} \right] \\ &= 150,000(3.629895) \\ &= \text{Php } 544,484.28 \text{ to the nearest centavos} \end{aligned}$$

Periodic Payment in an Ordinary Annuity

If the present value or the amount of an ordinary annuity is known, the periodic payment can be determined by solving the annuity formulas for R . hence, we have

$$R = \frac{Si}{(1+i)^n - 1} \quad \text{and} \quad R = \frac{iA_n}{1 - (1+i)^{-n}}$$

Example:

An ordinary annuity payable quarterly at 15% compounded every 3 months for 5 years and 9 months has a present value Php 125,000. How much is the quarterly payment?

Solution:

The quarterly payment is R . We know that

$$j = 0.15 \quad m = 4 \quad t = 5\frac{9}{12} = 5\frac{3}{4} \quad A_n = 125,000$$

Getting the value of i and n , it follows

$$i = \frac{j}{m} = \frac{0.15}{4} = 0.0375 \qquad n = t \cdot m = (5\frac{3}{4})(4) = 23$$

Hence, we have

$$A_n = R \left[\frac{1 - (1 + i)^{-n}}{i} \right]$$

It follows that

$$R = \frac{iA_n}{1 - (1 + i)^{-n}}$$

$$R = \frac{(0.0375)(125,000)}{1 - (1 + 0.0375)^{-23}}$$

$$R \approx \text{Php } 8,206.67$$



For more knowledge about Ordinary Annuity, please check the link provided;
<https://courses.lumenlearning.com/boundless-finance/chapter/annuities/>
<https://thismatter.com/money/investments/present-value-future-value-of-annuity.htm>

REMEMBER



- An **annuity** is a contract between you and an insurance company in which you make a lump-sum payment or series of payments and, in return, receive regular disbursements, beginning either immediately or at some point in the future.

- **Future Value of an Ordinary Annuity**

$$S = R \cdot s_{n|i} = R \left[\frac{(1 + i)^n - 1}{i} \right]$$

- **Present Value of an Ordinary Annuity**

$$A_n = R \cdot a_{n|i} = R \left[\frac{1 - (1 + i)^{-n}}{i} \right]$$

- **Periodic Payment in an Ordinary Annuity**

$$R = \frac{Si}{(1+i)^n - 1} \quad \text{and} \quad R = \frac{iA_n}{1 - (1+i)^{-n}}$$



APPLICATION

ACTIVITY: Solve the given problem and show your complete solution.

How much should be deposited in an account every 6 months in order to have Php 3,600,000 in 10 years? Money accumulates at 12% compounded semiannually.



REFERENCES

<https://www.mathbootcamps.com/simple-interest-formula-and-examples/>
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