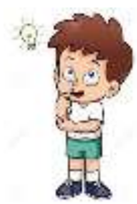
Chapter 2: Mathematical Language and Symbol

Objectives:
a. Understand the Mathematical Language by identifying the differences of Expression to Sentence.
b. Identify the four basic concepts of Mathematical language
c. Form a compound statement using connectives.

Lesson 1: Mathematical Language (Expression vs. Sentence)

A hypothetical situation



Imagine the following scenario: You're in math class, and the instructor passes a piece of paper to each student. It is announced that the paper contains Study Strategies for Students of Mathematics; you are to read it and make comments. Upon glancing at the paper, however, you observe that it is written in a foreign language that you do not understand!

The Importance Of Language

Is the instructor being fair? Of course not. Indeed, the instructor is probably trying to make a point. Although the ideas in the paragraph may be simple, there is no access to the ideas without a knowledge of the language in which the ideas are expressed. This situation has a very strong analogy in mathematics. People frequently have trouble understanding mathematical ideas: not necessarily because the ideas are difficult, but because they are being presented in a foreign language—the language of mathematics.

Characteristics of the Language of Mathematics

The language of mathematics makes it easy to express the kinds of thoughts that mathematicians like to express. It is:

- **Precise** (able to make very fine distinctions)
- **Concise** (able to say things briefly)
- **Powerful** (able to express complex thoughts with relative ease).

The language of mathematics can be learned, but requires the efforts needed to learn any foreign language. In this book, you will get extensive practice with mathematical language ideas, to enhance your ability to correctly read, write, speak, and understand mathematics.

Vocabulary Versus Sentences

Every language has its vocabulary (the words), and its rules for combining these words into complete thoughts (the sentences). Mathematics is no exception. As a first step in discussing the mathematical language, we will make a very broad classification between the 'nouns' of mathematics (used to name mathematical objects of interest) and the 'sentences' of mathematics (which state complete mathematical thoughts).

For example, inappropriately setting things equal to zero, or stringing things together with equal signs, as if '$=$' means 'I'm going on to the next step.' In the next few paragraphs, analogies between mathematics and English are explored; examples are presented to study these analogies; and finally the ideas are made more precise.

ENGLISH: Nouns Versus Sentences

In English, **nouns** are used to name things we want to talk about (like people, places, and things); whereas **sentences** are used to state complete thoughts. A typical English sentence has at least one noun, and at least one verb. For example, consider the sentence

Carol loves mathematics.

Here, 'Carol' and 'mathematics' are nouns; 'loves' is a verb.

MATHEMATICS: Expressions Versus Sentences

- **Expression**
  - An ***expression*** is the mathematical analogue of an English noun; it is a correct arrangement of mathematical symbols used to represent a mathematical object of interest.
  - An expression does ***not*** state a complete thought; it does not make sense to ask if an expression is ***true*** or ***false***.
  - The most common expression types are ***numbers***, ***sets***, and ***functions***.
  - Numbers have lots of different names: for example, the expressions

$$5 \quad 2+3 \quad 10 \div 2 \quad (6-2)+1 \quad 1+1+1+1+1$$

- **Sentence**
  - A mathematical ***sentence*** is the analogue of an English sentence; it is a correct arrangement of mathematical symbols that states a complete thought.

- Sentences have verbs.
  In the mathematical sentence '$3+4=7$', the verb is '$=$'.
- A sentence can be (always) true, (always) false, or sometimes true/sometimes false.

  For example,

  The sentence, '$1+2=3$' is true.
  The sentence, '$1+2=4$' is false.
  The sentence, '$x=2$' is sometimes true/sometimes false: it is true when x is 2, and false otherwise.
  The sentence, '$x+3=3+x$' is (always) true, no matter what number is chosen for x.

	ENGLISH	MATHEMATICS
name given to an object of interest:	NOUN (person, place, thing) Examples: Carol, Idaho, book	EXPRESSION Examples: 5, $2+3$, $\frac{1}{2}$
a complete thought:	SENTENCE Examples: The capital of Idaho is Boise. The capital of Idaho is Pocatello.	SENTENCE Examples: $3+4=7$ $3+4=8$

EXAMPLES:

2	is an expression
$1+1$	is an expression
$x+1$	is an expression
$1+1=2$	is a (true) sentence
$1+1=3$	is a (false) sentence
$x+1=3$	is a (sometimes true/sometimes false) sentence

So, x is to mathematics as ***cat*** is to English.

Conventions in Languages

Languages have conventions. In English, for example, it is conventional to capitalize proper names (like 'Carol' and 'Idaho'). This convention makes it easy for a reader to distinguish between a common noun (like 'carol', a Christmas song) and a proper noun (like 'Carol'). Mathematics also has its conventions, which help readers distinguish between different types of mathematical expressions. Listed below are the two common conventions in Mathematical Language.

1. When introducing a new variable into a discussion, the convention is to place the new variable to the left of the equal sign and the expression that defines it to the right.
   For example, in a computer program, if a and b have previously been defined, and you want to assign the value of $a + b$ to a new variable s, you would write something like $s = a + b$. Similarly, in a mathematical proof, if a and b have previously been introduced into a discussion, and you want to let s be their sum, instead of writing "$Let\ a + b = s$," you should write, "$Let\ s = a + b$."
2. It is considered good mathematical writing to avoid starting a sentence with a variable. That is one reason that mathematical writing frequently uses words and phrases such as Then, Thus, So, Therefore, It follows that, Hence, etc.

   For example, in a proof that any sum of even integers is even, instead of writing,
   By definition of even,
   $m = 2a\ and\ n = 2b$ for some integers a and b.
   $m + n = 2a + 2b \ldots$
   Write
   By definition of even, $m = 2a\ and\ n = 2b$ for some integers a and b.
   Then, $m + n = 2a + 2b \ldots$

   The fact that $m = 2a\ and\ n = 2b$ is a consequence of the facts that $m = 2a\ and\ n = 2b$. Including the word "Then" in your proof alerts your reader to this reasoning.

   Another Example: Sentence to convert into an expression

Verbal Phrase	Variable Expression
The sum of a number and 9	$n + 9$
The difference of a number and 21	$n - 21$
The product of 6 and a number	$6n$
The quotient of 48 and a number	$\frac{48}{n}$
One third of a number	$\frac{1}{3}n$

For more knowledge about Expression vs. Sentences, please check the link provided;
http://www.onemathematicalcat.org/algebra_book/online_problems/exp_vs_sen.htm
http://thinkmath.edc.org/resource/developing-mathematical-language

REMEMBER

The language of mathematics makes it easy to express the kinds of thoughts that mathematicians like to express. It is:

- **Precise** (able to make very fine distinctions)
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ACTIVITY: Expression vs. Sentence
Choose your partner, and make at least 10 sentences to convert into an expression. Use your knowledge to distinguish the difference of two.

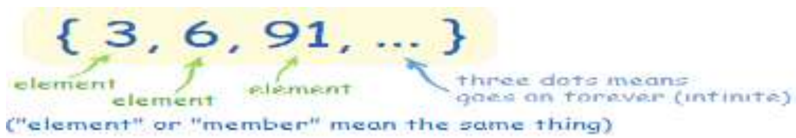
Lesson 2: Four Basic Concepts in Mathematical Language

(Set, Functions, Relation and Binary Operation)

- **Set**
  - Is a collection of distinct well-defined objects called *elements.*
  - Are denoted by upper case letter. If a is an element of set A, then we use the notation $a \in A$. Suppose b does not belong to set A, then we use the notation $b \notin A$.

Set Notation

There is a fairly simple notation for sets. We simply list each element (or "member") separated by a comma, and then put some curly brackets around the whole thing:



The curly brackets { } are sometimes called "set brackets" or "braces".
Examples:

Set of even numbers: $\{..., -4, -2, 0, 2, 4, ...\}$

Set of odd numbers: $\{..., -3, -1, 1, 3, ...\}$

Set of prime numbers: $\{2, 3, 5, 7, 11, 13, 17, ...\}$

Positive multiples of 3 that are less than 10: $\{3, 6, 9\}$

And the list goes on. We can come up with all different types of sets.

There can also be sets of numbers that have no common property, they are just **defined** that way. For example:

$\{2, 3, 6, 828, 3839, 8827\}$

$\{4, 5, 6, 10, 21\}$

$\{2, 949, 48282, 42882959, 119484203\}$

REFLECTION

Why are Sets Important?

Sets are the fundamental property of mathematics. Now as a word of warning, sets, by themselves, seem pretty pointless. But it's only when we apply sets in different situations do they become the powerful building block of mathematics that they are.

Math can get amazingly complicated quite fast. Graph Theory, Abstract Algebra, Real Analysis, Complex Analysis, Linear Algebra, Number Theory, and the list goes on. But there is one thing that all of these share in common: ***Sets***.

- **Relation**
  - is a set of ordered pairs. The set of all first components of the ordered pairs is called the *domain* of the relation, the set of all the second components are called the *range* of the relation.

  Example: Any of the following are then relations because they consist of a set of ordered pairs.

  $$\{(-2,5),(-1,0),(2,-3)\}$$
  $$\{(-1,0),(0,-3),(2,-3),(3,0),(4,5)\}$$
  $$\{(3,0),(4,5)\}$$
  $$\{(-2,5)(-1,0)(0,-3)(1,-4)(2,-3)(3,0)(4,5)\}$$

- **Function**
  - is a relation for which each value from the set the first components of the ordered pairs is associated with exactly one value from the set of second components of the ordered pair.

  Example #1: The following relation is a function.

  $$\{(-1,0),(0,-3),(2,-3),(3,0),(4,5)\}$$

  Solution:

  From these ordered pairs we have the following sets of first components (*i.e.* the first number from each ordered pair) and second components (*i.e.* the second number from each ordered pair).

  $1st\ components\ : \{-1,0,2,3,4\}$ and $2nd\ components\ : \{0,-3,0,5\}$

For the set of second components notice that the "-3" occurred in two ordered pairs but we only listed it once.

To see why this relation is a function simply pick any value from the set of first components. Now, go back up to the relation and find every ordered pair in which this number is the first component and list all the second components from those ordered pairs. The list of second components will consist of exactly one value.

For example, let's choose 2 from the set of first components. From the relation we see that there is exactly one ordered pair with 2 as a first component, $(2,-3)$). Therefore, the list of second components (*i.e.* the list of values from the set of second components) associated with 2 is exactly one number, -3.

Note that we don't care that -3 is the second component of a second ordered par in the relation. That is perfectly acceptable. We just don't want there to be any more than one ordered pair with 2 as a first component.

We looked at a single value from the set of first components for our quick example here but the result will be the same for all th e other choices. Regardless of the choice of first components there will be exactly one second component associated with it.

Therefore, this relation is a function.

Example #2: The following relation is not a function.

$$\{(6,10),(-7,3),(0,4),(6,-4)\}$$

Solution:

Don't worry about where this relation came from. It is just one that we made up for this example.

Here is the list of first and second components

$1st\ components$: $\{6,-7,0\}$ and $2nd\ components$: $\{10,3,4,-4\}$

From the set of first components let's choose 6. Now, if we go up to the relation we see that there are two ordered pairs with 6 as a first component : $(6,10)$ and $(6,-4)$. The list of second components associated with 6 is then : $10,-4$

The list of second components associated with 6 has two values and so this relation is not a function.

Note that the fact that if we'd chosen -7 or 0 from the set of first components there is only one number in the list of second components associated with each. This doesn't matter. The fact that we found even a single value in the set of first components with more than one second component associated with it is enough to say that this relation is not a function.

As a final comment about this example let's note that if we removed the first and/or the fourth ordered pair from the relation we would have a function!

- **Binary Operation**
  - is simply a rule for combining two values to create a new value. The most widely known binary operations are those learned in elementary school: addition, subtraction, multiplication and division on various sets of numbers.
  - A **binary operation** on a set is a calculation involving two elements of the set to produce another element of the set.

Situation 1:
It is possible to define "new" binary operations. Consider this example:

A new math (binary) operation, using the symbol Φ , is defined to be
$a\Phi b = 3a + b,$
where *a* and *b* are real numbers.

Question	Explanation
1. What is $8\ \Phi 3$?	Substitute the values of $a\ and\ b$ into the right-hand side of the definition, namely $3a + b$. $8\ \Phi\ 3 = 3 \bullet 8 + 3 = 24 + 3 = 27$
2. Is $a\Phi b$ commutative?	Does $a\Phi b = b\Phi a$ for all possible values? $3a + b = 3b + a$? Not true for all real numbers. If $a = 8$ and $b = 2$; $p :: q :$; $p \wedge q$; $26 \neq 14$. The operation Φ is not commutative for real numbers.
3. Is $a\Phi b$ associative?	Does $a\Phi\ (b\Phi c) = (a\Phi b)\Phi c$? $a\Phi\ (3b + c) = (3a + b)\ \Phi c$? $3a + (3b + c) = 3(3a + b) + c$? Not true for all reals. If $a = 2, b = 3,\ c = 4$; $3 \bullet 2 + (3 \bullet 3 + 4) \neq 3(3 \bullet 2 + 3) + 4$; $6 + 13 \neq 3(9) + 4$; $19 \neq 31$. The operation Φ is not associative for real numbers.

Situation 2:
Sometimes, a binary operation on a finite set (a set with a limited number of elements) is displayed in a table which shows how the operation is to be performed.

A binary operation, *, is defined on the set {1, 2, 3, 4}. The table at the right shows the 16 possible answers using this operation.
To read the table: read the first value from the left hand column and the second value from the top row. The answer is the intersection point.

*	1	2	3	4
1	4	3	2	1
2	3	1	4	2
3	2	4	1	3
4	1	2	3	4

Question	Explanation
1. What is $2 * 4$?	$2 * 4 = 2$ (where the row and column intersect)
2. Is * commutative?	Check: $3 * 1 = 1 * 3$, *yes*, $2 = 2$. Unfortunately, you now need to check all of the other possibilities. There is, however, a shorter way ... | * | 1 | 2 | 3 | 4 | | 1 | 4 | 3 | 2 | 1 | | 2 | 3 | 1 | 4 | 2 | | 3 | 2 | 4 | 1 | 3 | | 4 | 1 | 2 | 3 | 4 | draw a diagonal line from the upper left corner to the lower right corner of the table. If the table is symmetric with respect to this line, the table is commutative.
3. What is the identity element for the operation *?	Find the single element that will always return the original value. The identity element is 4. You will have found the identity element when all of the values in its row and its column are the same as the row and column headings.
4. Is * associative for these values? $4 * (3 * 2) = (4 * 3) * 2$	Does $4 * (3 * 2) = (4 * 3) * 2$? $4 * (4) = (3) * 2$ $4 = 4$ YES, this example is associative.

Take Note: Unfortunately, if you were asked the general question, "Is * associative?", instead of just checking one single case as shown in #4, you would have to check **ALL** possible arrangements. Unlike the commutative property, there is NO shortcut for checking associativity when working with a table. But remember, it only takes one arrangement which does not work to show that associativity fails.

For more knowledge about Basic Concepts of Mathematical Language, please check the link provided;
MathBitsNotebook.com

REMEMBER

- **Set i**s a collection of distinct well-defined objects called *elements.*
- **Relation** is a set of ordered pairs.
- **Function** is a relation for which each value from the set the first components of the ordered pairs is associated with exactly one value from the set of second components of the ordered pair.
- **Binary Operation** is simply a rule for combining two values to create a new value.

Direction: Answer the following with brief explanation.

A. Determine whether the set of ordered pairs represents a function or nonfunction.
   1. $\{(0,0),(1,1),(2,2)\}$
   2. $\{(1,-1),(1,-2),(1,-3)\}$
   3. $\{(15,-10),(12,-2),(1,-3)\}$

B. Answer the following using the Binary Operation.
   4. What is $5\,\Phi 3$?
   5. What is $4\,\Phi 3$?

Lesson 3: Elementary Logic

Propositional logic is a mathematical system for reasoning about propositions and how they relate to one another.

- Every statement in propositional logic consists of propositional variables combined via logical connectives.
- Each variable represents some proposition, such as "You liked it" or "You should have put a ring on it."
- Connectives encode how propositions are related, such as "If you liked it, then you should have put a ring on it."

Example: "Every square is a rhombus or every square is a parallelogram."

Propositional Variables

- A variable that represents propositions
- Propositional variables are usually represented as lower-case letters, such as $p,\ q,\ r,\ s,\ etc.$
- Each variable can take one of two values: true or false.
- Propositional variables in logic play the same role as numerical variables in arithmetic.

Example: "Every square is a rhombus or every square is a parallelogram."

Let $p =$ Every square is a rhombus
$q =$ every square is a parallelogram

We use the variable p and q to let the two proposition in a given statement.

Logical Connectives

- The logical connectives are defined by truth tables (but have English language counterparts).
- In addition to propositional variables, we have logical connectives such as not, and, or, conditional, and biconditional

Logic	*Math*	*English*
Conjunction	$\wedge$	and
Disjunction	$\vee$	or (inclusive)
Negation	~	not
Conditional	$\Rightarrow$	if ... then ...
Biconditional	$\Leftrightarrow$	if and only if

How to read Logical Connectives

- Logical NOT: $\sim p$
  - Read "not p"
  - $\sim p$ is true if and only if p is false.
  - Also called logical negation.

  Example: $p =$ Indonesia is in Asia.
  $\sim p =$ Indonesia is **not** in Asia.

- Logical AND: $p \wedge q$
  - Read "p and q."
  - $p \wedge q$ is true if both p and q are true.
  - Also called logical conjunction.

  Example: "Philippines is in Asia **and** the capital of Philippines is Manila."

- Logical OR: $p \vee q$
  - Read "p or q."
  - $p \vee q$ is true if at least one of p or q are true (inclusive OR)
  - Also called logical disjunction.

  Example: "Every square is a rhombus **or** every square is a parallelogram."

- Logical IMPLIED (IF…, THEN…): $p \rightarrow q$
  - Read "p implies q" or "if p …, then q …"
  - p and q might both be true.
  - p might be true and q false.
  - p might be false and q true.
  - p and q might both be false.
  - Also called logical implication.

  Example: "**If** Helen finishes her homework, **then** she will clean her room."

- Logical EQUIVALENT: $p \leftrightarrow q$ or $p \equiv q$
  - Read "p is equivalent to q" or "p is equivalent to q if and only if…)"
  - p and q are logically equivalent if they have identical truth values under all possible situations.
  - Also called logically equivalent.

  Example: "I will get wet **if and only if** it rains."

Quantifiers

- Are words that denote the number of objects or cases referred to in a given statement. It comes from the Latin word "*quantos*". English quantifiers include "all", "none", "some", and "not all".
- The quantifiers "all", "every", and "each" illustrate that each and every object or case satisfies the given condition.
- The quantifiers "some", "several", "one of" and "part of" illustrate that not all but at least one object or case satisfies the given condition.

Example:

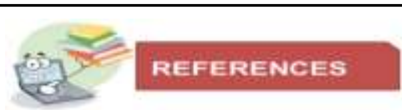
"All students are intelligent."
"Every student is intelligent."
"Each student is intelligent."
"Any student is intelligent."

For more knowledge about Elementary Logic, please check the link provided;
https://study.com/academy/lesson/symbolic-logic-definition-examples.html
https://study.com/academy/lesson/propositions-truth-values-and-truth-tables.html

REMEMBER

- **Propositional logic** is a mathematical system for reasoning about propositions and how they relate to one another.
- **Propositional Variables** is a variable that represents propositions
- **Logical Connectives** are the words defined by a truth table.
- **Quantifiers** denotes how many the cases given in a statement.

ACTIVITY: Make your own logic
Make 2 propositional logic for each logical connectives.



Soaring 21st Century of Mathematics, p.1
https://www.utm.edu/staff/jfieser/class/120/9-logic.htm